MODEL THEORY: AN EXPANDED SYLLABUS FOR ALG500009

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1. Suggested reading

- Marcja, A. and Toffalori, C.: *A Guide to Classical and Modern Model Theory*, Trends in Logic, Kluwer, 2003
- Hodges, W.: A shorter model theory

References starting with MT point to the first book. These notes form an expanded syllabus for the course ALG500009. The course and these notes are heavily based on the work of others. Besides the two above mentioned books, the following were used:

- Geschke, S.: Model Theory (lecture notes)
- Pillay, A.: Model Theory (lecture notes, autumn 2002)

Due to the nature of this text we do not provide references to standard notions and theorems.

2. Preliminaries

We assume the reader is familiar with First Order Logic (FOL). Formally, we work in ZF(C) and define:

Definition. A *language* $\mathscr{L}$ is a quadruple $\langle Const, Rel, Func, Var\rangle$ of disjoint sets, where $Const$ is the set of constant symbols of $\mathscr{L}$ (e.g. $0, 1, \emptyset, a, \dots$), Rel the set of relation symbols (e.g. $\in, \leq, R, \dots$), $Func$ the set of function symbols (e.g. $+, \cdot, f, \dots$) and Var the (usually countable) set of variable names. Each relation and function symbol s has an associated natural number $Ar(s)$, called the *arity* of s.

Note. For some languages, one or more of the above sets (usually excepting Var) may be empty.

Definition. Given a language $\mathscr{L}$, an *atomic term* of this language is any constant or variable symbol. The set of *$\mathscr{L}$-terms* is the smallest set containing all atomic terms and closed under:

(1) If f is a an n-ary function symbol and $t_1, \dots, t_n$ are a terms then so is $f(t_1, \dots, t_n)$.

A term is called *ground* if it does not contain any variables.

Similarly, the set of *atomic $\mathscr{L}$-formulas* is defined as:

(1) If t_1, t_2 are terms, then $t_1 = t_2$ is an atomic formula; and
(2) If R is an n-ary relation symbol and $t_1, \dots, t_n$ are terms, then $R(t_1, \dots, t_n)$ is an atomic formula.

and an *$\mathscr{L}$-formula* is a sequence of symbols from the set $\{\exists, \vee, \neg, =, (,)\} \cup \mathscr{L}$ defined in the usual inductive way, i.e. as the smallest set containing all atomic formulas and closed under:

(1) If φ, ψ are formulas then so are $(\varphi \vee \psi)$, $\neg\varphi$; and
(2) If φ is a formula and x is a variable, then $\exists x\varphi$ is a formula.

We denote the set of all $\mathscr{L}$-formulas by $Fml(\mathscr{L})$. Given a formula φ and a variable $x \in Var$, we inductively define whether an occurance of x in φ is *free*: Any occurance of x in an atomic formula is free. If φ is $\varphi_1 \vee \varphi_2$, then any free occurance of x in φ_1 or φ_2 is also a free occurance in φ; any occurance of x in $\exists x\varphi$ is not free. An occurance of x that is not free is called *bound*. A variable that

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has a free occurance in φ is called a free variable of φ. A formula without free variables is called a *sentence*. We write $\varphi(x_0,\dots,x_n)$ to mean that φ is a formula whose free variables are $\{x_0,\dots,x_n\}$.

Note. Formally, $\&,\forall,\to,\leftrightarrow$ are not allowed as symbols in formulas. However we take them, with their usual meaning, to be convenient shorthand, e.g. $\forall x\varphi$ is a shorthand for $\neg\exists x\neg\varphi$.

Note. If the set of variables is infinite we may, by a suitable renaming of variables, always guarantee that either all occurrances of any variable in a formula φ are bound or they are all free.

Note. In this course we will only consider finite formulas. In general, it is possible to define formulas in a way that allows infinite strings (e.g. infinite conjunctions). Another generalization is to allow different quantifiers (e.g. exists uncountably many, or Henkin's branching quantifiers). However, in these generalizations, some familiar facts fail to hold. In fact, Lindström (see Lindstrom, P.: *First order predicate logic with generalized quantifiers* ,Theoria 32 (1966), 186-195; or J. Vaananen's short *sketch*), proved that this is necessary, i.e. classical first-order logic is the strongest logic for which the compactness and Löwenheim-Skolem theorems hold.

Now that we have defined the syntax of FOL, we want to define its semantics. We first define what a structure for a language $\mathscr{L}$ is:

Definition. If $\mathscr{L} = (Const, Rel, Func, Var)$ is a language, a *structure* for $\mathscr{L}$ (or an $\mathscr{L}$-structure) is a tuple $\mathscr{A} = \langle A, C^A, R^A, F^A : C \in Const, R \in Rel, F \in Func\rangle$ where

(1) A is a nonempty set, the *universe* of $\mathscr{A}$;
(2) C^A is an element of A for each constant symbol $C \in Const$;
(3) $R^A \subseteq A^{Ar(R)}$ is an $Ar(R)$-ary relation on A for each relation symbol $R \in Rel$; and
(4) $F^A : A^{Ar(F)} \to A$ is a function of $Ar(F)$ variables for each function symbol $F \in Func$.

If S is a symbol of $\mathscr{L}$ we call S^A the *interpretation* of S in $\mathscr{A}$.

Note. If there is no danger of confusion, we will often write A instead of $\mathscr{A}$ and vice versa. Also note that we require A to be nonempty. This is just for technical reasons to avoid some pathologies.

Note. We could have omitted constant symbols in the above definition without loosing anything (a constant symbol can be thought of just as a function symbol of arity 0). Also note that function symbols cannot be thought of as a special case of relation symbols (or vice versa), since, we require their interpretations to be *total* functions. We will see that this makes languages without function symbols decidedly different than languages with them. Most of our languages will have some relation symbols; model theory studying languages without relation symbols is called universal algebra ;-)

We now proceed to tie formulas and structures together by defining the satisfaction relation. The following is the standard definition for FOL:

Definition. Let $\mathscr{A}$ be a structure for some language $\mathscr{L}$. A *valuation* is any function $v : Var^{\mathscr{L}} \to A$. Given a term t and a valuation v we inductively define the interpretation $t^{\mathscr{A}}[v]$ of t in $\mathscr{A}$:

(1) If t is a variable x, then $t^{\mathscr{A}}[v] = v(x)$;
(2) if t is a constant c, then $t^{\mathscr{A}}[v] = c^A$; and
(3) if t is of the form $f(t_1,\dots,t_n)$, then $t^{\mathscr{A}}[v] = f(t_1^{\mathscr{A}}[v],\dots,t_n^{\mathscr{A}}[v])$.

Note. If t is a ground term, then $t^{\mathscr{A}}[v]$ does not depend on the valuation and we will write just $t^{\mathscr{A}}$.

Definition. Next, given a formula φ and a valuation v, we inductively define what it means that $\mathscr{A} \models \varphi[v]$, i.e. that $\mathscr{A}$ *satisfies* $\varphi[v]$:

(1) If φ is $t_1 = t_2$ then $\mathscr{A} \models \varphi[v]$ if $t_1^{\mathscr{A}}[v] = t_2^{\mathscr{A}}[v]$;
(2) if φ is $R(t_1,\dots,t_n)$ then $\mathscr{A} \models \varphi[v]$ if $R^{\mathscr{A}}(t_1^{\mathscr{A}}[v],\dots,t_2^{\mathscr{A}}[v])$;
(3) if φ is $\varphi_1 \vee \varphi_2$ then $\mathscr{A} \models \varphi[v]$ if $\mathscr{A} \models \varphi_1[v]$ or $\mathscr{A} \models \varphi_2[v]$; similarly if φ is $\neg\varphi_1$ then $\mathscr{A} \models \varphi[v]$ if $\mathscr{A} \not\models \varphi_1[v]$; and
(4) if φ is $\exists x\psi$ then $\mathscr{A} \models \varphi[v]$ if there is a valuation v' which agrees with v on all variables except possibly x and $\mathscr{A} \models \psi[v']$.

If φ is a sentence, then it is easy to see that satisfaction does not depend on the valuation and we write just $\mathscr{A} \models \varphi$. Also, if $\varphi(x_0, \dots, x_n)$ is a formula and $\langle a_0, \dots, a_n \rangle \in A^n$ we write $\mathscr{A} \models \varphi(a_0, \dots, a_n)$ to mean that $\mathscr{A} \models \varphi[v]$ for any valuation v such that $v(x_0) = a_0, \dots, v(x_n) = a_n$.

Note. Given a structure $\mathscr{A}$ for a language $\mathscr{L}$ and a set $X \subseteq A$ we can enlarge the language $\mathscr{L}$ to a language $\mathscr{L}(X)$ by adding a new constant c^x for each $x \in X$. Then the structure $\mathscr{A}$ naturally expands into a structure $\mathscr{A}_X$ for $\mathscr{L}(X)$ where each new constant c^x is interpreted as x. If $\varphi \in Fml(\mathscr{L}(X))$ we will abuse notation and write $\mathscr{A} \models \varphi$ to mean $\mathscr{A}_X \models \varphi$.

Note. To state the above quoted Lindström's theorem, one would need to generalize the above concepts and introduce a general satisfaction relation between abstract sentences and structures. This relation would still need to satisfy some variant of 3 and 4 above. For more information see, e.g. Barwise, J. and Feferman, S (eds): *Model-theoretic logics*, Perspectives in Mathematical Logic. Springer-Verlag, New York, 1985.

2.1. **Morphisms.** Now that we have defined the basic objects we will be concerned with (structures), we proceed in the best tradition of category theory to define morphisms:

Definition. Given $\mathscr{A}$ and $\mathscr{B}$ two structures for a language $\mathscr{L}$ we say that a function $f : A \to B$ is a *morphism* of structures if it "preserves" the language, i.e.

(1) $f(c^{\mathscr{A}}) = c^{\mathscr{B}}$ for each constant $c \in \mathscr{L}$;
(2) $f(F^{\mathscr{A}}(a_1, \dots, a_n)) = F^{\mathscr{B}}(f(a_1), \dots, f(a_n))$ for each n-ary function symbol $F \in \mathscr{L}$; and
(3) if $R^{\mathscr{A}}(a_1, \dots, a_n)$ then $R^{\mathscr{B}}(f(a_1), \dots, f(a_n))$ for each n-ary relation symbol $R \in \mathscr{L}$.

If f is an injection and additionally satisfies:

(1) if $R^{\mathscr{B}}(f(a_1), \dots, f(a_n))$ then $R^{\mathscr{A}}(a_1, \dots, a_n)$ then

we say that it is an *embedding*. An embedding which is onto B is called an *isomorphism* (we write $f : \mathscr{A} \simeq \mathscr{B}$). We say that $\mathscr{A}$ is a substructure of $\mathscr{B}$ (and write $\mathscr{A} \leq \mathscr{B}$) if $A \subseteq B$ and the inclusion map $\subseteq : A \to B$ is a morphism.

Note. If $\mathscr{B}$ is a structure for a language without function symbols, then any subset $A \subseteq B$ containing realizations of all the constant symbols is naturally a substructure (the realization of relation symbols will be just their restriction to A). This, however, fails if our language contains function symbols.

Exercise. Suppose $\mathscr{A}, \mathscr{B}$ are two substructures of some structure $\mathscr{C}$ in some language. Then $\mathscr{A} \cap \mathscr{B}$ is again a substructure of $\mathscr{C}$.

Exercise. An increasing union of substructures of some structure $\mathscr{C}$ is again a substructure of $\mathscr{C}$

The following exercise gives a characterization of isomorphisms with a more logical flavour to it:

Exercise (MT:1.3.2)**.** Suppose $\mathscr{A}, \mathscr{B}$ are structures for some language $\mathscr{L}$ and $f : A \to B$ is a bijection of A onto B. Show that f is an isomorphism iff for every formula $\varphi(\bar{x}) \in Fml(\mathscr{L})$ and every sequence $\bar{a} \in \mathscr{A}$ of elements of A we have $\mathscr{A} \models \varphi(\bar{a}) \leftrightarrow \mathscr{B} \models \varphi(f(\bar{a}))$.

The above exercise motivates the definition of an elementary embedding:

Definition. An embedding $f : \mathscr{A} \to \mathscr{B}$ is *elementary* if for every formula $\varphi(\bar{x}) \in Fml(\mathscr{L})$ and every sequence $\bar{a} \in \mathscr{A}$ of elements of A we have $\mathscr{A} \models \varphi(\bar{a}) \leftrightarrow \mathscr{B} \models \varphi(f(\bar{a}))$. If $\subseteq$ is an elementary embedding of $\mathscr{A}$ into $\mathscr{B}$ then we say that $\mathscr{A}$ is an *elementary substructure* of $\mathscr{B}$ ($\mathscr{A} \preceq \mathscr{B}$). The embedding f is called *existential* if it satisfies this condition for existential (Σ_1) formulas (and, in case of $\subseteq$, we write $\mathscr{A} \preceq_{\Sigma_1} \mathscr{B}$).

Note. The above definition can be extended to functions $f : X \to \mathscr{B}$, i.e. functions whose domain need not necessarily be a structure. In this case we shall say that f is an elementary map or an existential map.

Exercise. Show that an elementary map is necessarily one-to-one and its inverse is also elementary.

Theorem (Tarski-Vaught test,MT:1.3.18)**.** *A substructure $\mathscr{A} \le \mathscr{B}$ is an elementary substructure iff for every formula $\varphi(x)$ of $\mathscr{L}(A)$ we have that if $\mathscr{B} \models (\exists x)\varphi$ then also $\mathscr{A} \models (\exists x)\varphi$.*

Definition. A *partial isomorphism* of structures $\mathscr{A}$ and $\mathscr{B}$ is an isomorphism between two substructures. A *back-and-forth system* for $\mathscr{A}$ and $\mathscr{B}$ is a (nonempty) family $\mathscr{I}$ of partial isomorphisms such that for each $f \in \mathscr{I}$ and each $a \in A, b \in B$ there is $g \in \mathscr{I}$ extending f with $a \in \operatorname{dom} g$ and $b \in \operatorname{rng} g$. Two structures are said to be *partially isomorphic* ($\mathscr{A} \simeq_p \mathscr{B}$) if they have a back-and-forth system.

Definition. Two structures $\mathscr{A}$ and $\mathscr{B}$ are *elementarily equivalent* ($\mathscr{A} \equiv \mathscr{B}$) if they satisfy the same sentences.

Theorem. $\mathscr{A} \simeq B \to \mathscr{A} \simeq_p B \to \mathscr{A} \equiv B$.

Proof. Only the second implication needs some work. See MT:1.3.10. □

As the examples below show, none of the implications can be reversed in general. However, for countable structures we have:

Theorem (MT:1.3.8)**.** *Two countable partially isomorphic structures are isomorphic.*

Example. $\mathbb{R} + \mathbb{Q} \simeq_p \mathbb{Q}$ (as structures for $\{\le\}$) while, of course, $\mathbb{R} + \mathbb{Q} \not\simeq \mathbb{Q}$

Example. A nonstandard model of $Th(\mathbb{N})$ cannot be partially isomorphic to $\mathbb{N}$.

Theorem (Fraïssé,Mt:1.3.11)**.** *Two structures $\mathscr{A}, \mathscr{B}$ in a finite language $\mathscr{L}$ are elementarily equivalent if there is a decreasing sequence $\langle \mathscr{I}_n : n < \omega \rangle$ of non empty sets $\mathscr{I}_n$ of partial isomorphisms between $\mathscr{A}$ and $\mathscr{B}$ such that for each $f \in \mathscr{I}_{n+1}$ and each $a \in A, b \in B$ there is $g \in \mathscr{I}_n$ extending f such that $a \in dom\ g$ and $b \in rng\ g$.*

2.2. Basic tools.

Theorem (Compactness)**.** *If T is a set of sentences in some language $\mathscr{L}$ such that for each finite $T' \subseteq T$ there is a model $\mathscr{A}^{T'}$ of T' then there is a model of T.*

Theorem (Löwenheim-Skolem)**.** *If $\mathscr{A}$ is an infinite structure for a language $\mathscr{L}$ with $|\mathscr{L}| = \kappa$ then for every $\lambda \ge \kappa$ there is a structure $\mathscr{B}$ of size λ elementarily equivalent to $\mathscr{A}$.*

2.3. The goal: Morley's theorem.

Definition. A class of structures $\mathscr{K}$ for some language is *elementary* if it is closed under elementary equivalence and ultraproducts.

Example. (the non-examples are all based on compactness)

- infinite structures from an elementary class while finite ones do not!
- dense linear orders are elementary while (order)complete dense linear orders are not!
- wellorders are not elementary

Definition. A set of sentences T of some language $\mathscr{L}$ is called a *theory*. We say that $T \models \varphi$ if for every model $\mathscr{A}$ of T we have $\mathscr{A} \models \varphi$. (A structure $\mathscr{A}$ is a model of T if it satisfies each sentence from T). A theory is *complete* if for each sentence $\varphi \in Fml(\mathscr{L})$ either $T \models \varphi$ or $T \models \neg\varphi$.

Definition. A theory T is λ-categorical if there is, up to isomorphism, only a single model of T of cardinality λ.

Theorem (Vaught,1.6.4)**.** *If a theory T in a language of size at most λ has no finite models and is λ-categorical then it is complete.*

Theorem (Morley)**.** *If T is categorical in some* uncountable *cardinality, then it is categorical in* all uncountable *cardinalities.*

3. Definable Sets & Types

Definition. If $\mathscr{A}$ is a structure for a language L and $P \subseteq A$ we say that a set $Y \subseteq A^n$ is *P-definable* (in $\mathscr{A}$) if there is a formula φ (with n free variables) of $L(P)$ such that

$$Y = \{\bar{a} \in A^n : \mathscr{A} \models \varphi(\bar{a})\}.$$

The set Y is *definable* (in $\mathscr{A}$) if it is A-definable. We shall denote by $\mathscr{B}_n(P,\mathscr{A})$ the collection of all P-definable subsets of A^n.

Example. Let $\mathscr{L} = \emptyset$. Then $\mathscr{B}_n(A,\mathscr{A})$ consists just of the finite and co-finite sets.

Observation. *Notice that the following is true:*

(1) *$\mathscr{B}_n(P,\mathscr{A})$ is a Boolean algebra;*
(2) *every finite and co-finite subset of A^n is in $\mathscr{B}_n(A,\mathscr{A})$.*
(3) *If $\mathscr{A}$ is infinite, then there is a subset of A^n which is not definable.*

Note (see MT:5.2.1)**.** The definable sets could also have been defined as the smallest subalgebra of $\mathscr{P}(\bigcup_{n<\omega} A^n)$ containing the graph of every relation (incl. equality), and function of L and closed under projections and fibers.

Example. Let L be the language of fields ($\{0,1,+,\cdot,-\}$) and $\mathscr{K}$ a field. An *algebraic curve* over $\mathscr{K}$ is a set of solutions (i.e. zero's) of some polynomial (possibly in multiple variables) in $\mathscr{K}$. An *algebraic variety* is an intersection of finitely many *algebraic curves*. Algebraic varieties form a sublattice of $B_n(K,\mathscr{K})$. However, they typically aren't closed under complement (e.g. an algebraic variety over $\mathbb{R}^n$ is a closed set (in the Euclidean topology), in particular its complement can be an algebraic variety iff it is trivial).

Example. Let $L = \{+,\cdot\}$ and consider the standard model $\mathbb{N}$ of arithmetic as a structure for L. Then all definable subsets of $\mathbb{N}$ are $\emptyset$-definable. Moreover, every recursively enumerable set is also definable.

Proof (hint): By induction show that every element of $\mathbb{N}$ is $\emptyset$-definable. For the rest of the proof, see recursion theory :-)

Exercise. Consider $\mathbb{N}, Z$ as structures for $L = \{+,\cdot\}$. Show that $\mathbb{N}$ is $\emptyset$-definable in $\mathbb{Z}$.

Hint. Use Lagrange's theorem.

One can also arrive at the definable sets in a purely syntactic way.

Definition. Let $Fml_n(L)$ denote the set of formulas of L with n free variables. Given theory T in L let $\varphi \simeq_T \psi$ if $T \vdash \varphi \leftrightarrow \psi$. If X is a set of elements, we write just $Fml_n(X)$ instead of $Fml_n(L(X))$ if L is clear from the context.

Observation. *If T is a theory in L then $Fml_n/\simeq_T$ is a Boolean algebra.*

3.1. **Observation.** *$\mathscr{B}_n(X,\mathscr{A})$ is isomorphic to $Fml_n(X)/\equiv_{\mathscr{A}}$, where $\varphi \equiv_{\mathscr{A}} \psi$ if $Th(\mathscr{A})$ proves $\varphi \leftrightarrow \psi$. Denote the algebra $Fml_n(X)/\equiv_{\mathscr{A}}$ as $\mathscr{L}(X,\mathscr{A})$.*

Definition (MT:5.3)**.** A *complete n-type* over X is an ultrafilter on $\mathscr{B}_n(X,\mathscr{A})$. We denote by $S_n(X,\mathscr{A})$ the set of complete n-types over X.

Note. By observation 3.1, if T is a complete theory, then the complete n-types do not really depend on the structure $\mathscr{A}$ but rather on the theory $Th(\mathscr{A})$ of the structure. In this case we may look at a (complete) n-type over X as on a (maximal) set of consistent formulas of $L(X)$ closed under conjunction and provable implications. Also, it makes sense to talk about types over a set without mentioning the structure $\mathscr{A}$.

Definition (5.3.1,5.3.2)**.** A *complete n-type* over X is a maximal filter on the algebra $\mathscr{L}(X,\mathscr{A})$

3.2. **Topological interlude.**

Definition. A *topological space* is a set X of points together with a topology $\tau \subseteq \mathscr{P}(X)$ (i.e. the family of open sets) such that

- $\emptyset, X \in \tau$ (the empty set and the whole space are open sets)
- if $U, V \in \tau$ then $U \cap V \in \tau$ (a finite intersection of open sets is open)
- if $\mathscr{U} \subseteq \tau$ then $\bigcup \mathscr{U} \in \tau$ (a union of open sets is open)

A subset of X is closed if its complement is open (i.e. if its complement is in τ).

Definition. A family $\mathscr{B}$ of open subsets of X is a *(topological) basis* of X if it is closed under finite intersections and any open subset of X can be written as a union of sets from X.

Definition. A topological space is *Hausdorff* (or T_2) if for any two points $x, y \in X$ there are disjoint open sets U, V such that $x \in U$ and $y \in V$. (We say that U and V separate the points x, y.)

Definition. A topological space is *zero-dimensional* if it has a basis consisting of sets which are closed and open (*clopen* for short) at the same time.

Definition. A topological space X is *compact* if it is Hausdorff and any open cover of X has a finite subcover, i.e. if for any family $\mathscr{U}$ of open subsets of X such that $\bigcup \mathscr{U} = X$ — an open cover — there are finitely many elements $U_0, \dots, U_n \in \mathscr{U}$ — a finite subcover — such that $U_0 \cup \dots \cup U_n = X$.

Theorem. *A T_2 topological space X is compact iff any centered family $\mathscr{F}$ of closed sets (i.e. any finite number of elements of $\mathscr{F}$ have nonempty intersection) has a nonempty intersection.*

Theorem. *$S_n(X, \mathscr{A})$ together with the topology given by the basis consisting of sets of the form*

$$\hat{B} = \{p \in S_n(X, \mathscr{A}) : B \in p\},$$

where $B \in \mathscr{B}_n(X, \mathscr{A})$, is a compact zerodimensional space.

Proof. To see that the space is zerodimensional, notice that $\hat{B} = A \hat{\setminus} B$. To see that it is T_2, fix two distinct ultrafilters $p_1, p_2 \in S_n(X, \mathscr{A})$. Since they are distinct, we can choose, wlog, $B \in p_1 \setminus p_2$. Then $B \in p_1 \setminus p_2$ are two open sets separating p_1, p_2. To show that it is compact, let $\mathscr{U}$ be an open cover of the space and, wlog, assume it consists of basic open sets, i.e. $\mathscr{U} = \{\hat{B}_\alpha : \alpha \in \kappa\}$ for some κ. Since no finite subfamily $\{\hat{B}_\alpha : \alpha \in F\} \subseteq \mathscr{U}$ is a cover, for each such family there is a point $x_F \in A \setminus \bigcup_{\alpha \in F} B_\alpha$. This shows that the system $\{A \setminus \bigcup_{\alpha \in F} B_\alpha : F \subseteq \kappa \ \& \ |F| < \omega\}$ is a centered system of definable sets (the union of finitely many definable sets is definable). So we can extend it to an ultrafilter p. It is easy to see that $p \notin \bigcup \mathscr{U}$ contradicting the assumption that $\mathscr{U}$ was a cover. □

3.3. **Realizing types.**

We shall now investigate types. The crucial notion will be that of a *realized type*. We will see that models having many (or only a few) realized types will have interesting properties.

Definition. A model $\mathscr{A}$ *realizes* a type $p \in S_n(X, \mathscr{A})$ if there is some $a \in \bigcap p$.

The simplest examples of realized types are types generated by a point (or a tuple of points):

Definition. Given $\bar{a} \in A^n$ and $X \subseteq A$, the set

$$tp_{\mathscr{A}}(\bar{a}/X) = \{B \in \mathscr{B}_n(\mathscr{A}, X) : \bar{a} \in B\}$$

is a complete n-type over X, called the *type of $\bar{a}$* over X (or the *trivial type generated by $\bar{a}$*).

Exercise. Check that $tp_{\mathscr{A}}(\bar{a}/X)$ is really a complete n-type over X.

In a future section we will be interested in the possible ways of partitioning a given definable set into definable subsets. In some cases, this may not be possible (the trivial case being if the definable set is a singleton). This leads to the following definition

Definition. A type $p \in S_n(X, \mathscr{A})$ is *isolated* if it there is a (necessarily definable) set $B \in p$ which cannot be partitioned into proper definable subsets.

Exercise. Show that a type p is isolated iff there is a set $B \in p$ such that $\hat{B} = \{p\}$. (In general, a point in a topological space is isolated if it has a neibourghood consisting only of itself).

Observation. *An isolated type is always realized.*

Proof. Let $B \in p$ witness that p is isolated. Let $x \in B$. Since $tp_{\mathscr{A}}(x)$ is a complete type and $tp_{\mathscr{A}}(x) \in \hat{B}$ it follows, by the previous exercise, that $tp_{\mathscr{A}}(x) = p$. But then $x \in \bigcap p$. □

Note that an isolated type can be reconstructed from the single definable set which witnesses that it is isolated, i.e. if $\{p\} = \hat{B}$ then $p = \{C \in \mathscr{B}_n(X, \mathscr{A}) : B \subseteq C\}$. So, in some sense, it is rather simple. We shall later need an even stronger notion of "simple", given by the following definition:

Definition. A type $p \in S_n(X, \mathscr{A})$ is *algebraic* if there is a finite $B \in p$.

Observation. *An algebraic type is isolated.*

Example (5.3.9)**.** Consider some infinite set A as a structure for an empty language and let $X \subseteq A$ be a finite subset of A. Then all elements of $A \setminus X$ realize the same type over X. This type is isolated (as X is finite) but is not algebraic.

However, if we allow sufficiently many parameters, the implication can be reversed:

Exercise. If $p \in S_n(A, \mathscr{A})$ is isolated, then it is algebraic.

Before we move on to analyzing models based on the types they have, let us mention a simple observation:

Observation. *If the language is countable, then* $|X| \leq |S_n(X, \mathscr{A})| \leq 2^{\aleph_0 \cdot |X|}$.

Later on we shall see that the size of $S_n(X, \mathscr{A})$ is a strong indication of the "complexity" of $Th(\mathscr{A})$.

4. Atomic models, Saturated models

In this section we shall look a two kinds of models which lie on the opposite extremes when considering the types they realize. First we shall look at saturated models, which realize as many types as possible. Then we shall consider prime models which, conversely, realize as few types as possible.

4.1. **Saturated models.**

Definition (5.4.1)**.** A model $\mathscr{A}$ is *λ-saturated* if it realizes every 1-type over all subsets of cardinality $< \lambda$. It is *saturated* if it is $|\mathscr{A}|$-saturated.

Exercise. Show that a model cannot be $|A|^+$-saturated.

Exercise. Show that if a model is λ-saturated, then it realizes every n-type over all subsets of cardinality $< \lambda$.

It might not be clear at first sight whether saturated models exist. So lets start with a prototypical example:

Example. The rational numbers $(\mathbb{Q}, \leq)$ are *saturated*.

Proof. Notice that given $x_0, \ldots, x_n \in \mathbb{Q}$ the algebra $\mathscr{B}(\{x_0, \ldots, x_n\}, \mathbb{Q})$ consists of finite unions of intervals whose endpoints are in the set $\{x_0, \ldots, x_n\}$. So the algebra is finite. It follows that each type over a finite set is isolated and hence realized. □

Notice that $\mathbb{Q}$ is universal among countable linear orders in the sense that any other countable linear order is isomorphic to a subset of $\mathbb{Q}$. This is not true for all saturated models, but we do have something slightly weaker:

4.2. **Theorem** (5.4.5)**.** *If $\mathscr{A}$ is a λ-saturated model of a complete T and $\mathscr{B}$ is a model of T of cardinality $< \lambda$ then $\mathscr{B}$ can be elementarily embedded into $\mathscr{A}$.*

Proof. Let $\kappa = |B| < \lambda$ and enumerate B as $\{b_\alpha : \alpha < |B|\}$. We shall recursively construct a sequence of elementary functions $\langle f_\alpha : \alpha < \kappa\rangle$ satisfying:

(1) $f_\alpha \subseteq f_\beta$ for each $\alpha < \beta < \kappa$;
(2) $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$ for limit $\alpha < \kappa$; and
(3) $b_\alpha \in \mathrm{dom}(f_{\alpha+1})$ for each $\alpha < \kappa$.

Let $f_0 = \emptyset$. This is elementary since $\mathscr{B} \equiv \mathscr{A}$ by assumption. On limit steps we just take unions to satisfy (2). So let f_α be constructed. If $b_\alpha \in f_\alpha$ then we can let $f_{\alpha+1} = f_\alpha$. Otherwise, let $p \in S(\{b_\beta : \beta < \alpha\}, \mathscr{B})$ be the complete type of b_α over $\{b_\beta : \beta < \alpha\}$ in $\mathscr{B}$. Since f_α is elementary, the image $f_\alpha[p]$ can be extended to a complete type $q \in S(\{f(b_\beta) : \beta < \alpha\}, \mathscr{A})$. Since $\mathscr{A}$ is saturated (and since $\alpha < \kappa < \lambda$), this type is realized, say by some a_α. We can then extend f_α to $f_{\alpha+1}$ by letting $f_{\alpha+1}(b_\alpha) = a_\alpha$. It is routine to check that $f_{\alpha+1}$ will still be elementary. □

Another well known property of the rationals can be generalized to saturated models. Recall that Cantor showed that any two countable dense linear orders are isomorphic. Compare this with the following theorem:

Theorem (5.4.7)**.** *Any two elementary equivalent saturated models of the same cardinality are isomorphic.*

Proof. Proceed as in the proof of the previous theorem but ensure, using the fact that $\mathscr{B}$ is saturated, that the range of the functions covers the target model (this is the standard back-and-forth argument). Then recall that an elementary embedding onto a model is an isomorphisms. □

Let us now turn our attention to the problem of constructing general saturated models. We start with two simple lemmas:

Lemma. *If $\mathscr{A} \preceq \mathscr{B}$ and $X \subseteq A$ then $S_n(X, \mathscr{A}) \simeq S_n(X, \mathscr{B})$.*

An easy application of the compactness theorem (and the previous lemma) gives us:

4.3. **Lemma.** *If $p \in S(X, \mathscr{A})$ then there is an elementary extension of $\mathscr{A}$ of size $|\mathscr{A}|$ realizing p.*

Proof. Let c be a new constant not appearing in $\mathscr{L}_A$ and consider the theory $T = Th(\mathscr{A}) \cup \{\varphi(c) : \varphi(\mathscr{A}) \in p\}$ in the language $\mathscr{L}_A \cup \{c\}$. Since p is a type, this is a consistent theory. By the compactness theorem it has a model $\mathscr{B}$ and by the Löwenheim-Skolem theorem we can assume it has size $|\mathscr{A}$. It is not hard to see that the (reduct of) $\mathscr{B}$ is an elementary extension of $\mathscr{A}$ which realizes p. □

Theorem (5.4.2)**.** *If λ is an infinite cardinal, then every model $\mathscr{A}$ has a λ-saturated elementary extension.*

Proof of Claim. We shall only prove the case $\lambda = \aleph_0$.

Claim. Every model $\mathscr{A}$ has an elementary extension $\mathscr{A}^*$ which realizes all types over any subset of $\mathscr{A}$.

Proof of Claim. Enumerate all types over (a subset of) $\mathscr{A}$ as $\{p_\alpha : \alpha < \kappa\}$ and repeatedly use the previous lemma to build an elementary chain of models $\mathscr{A}_\alpha$ such that $\mathscr{A}_{\alpha+1}$ realizes p_α. Finally let $\mathscr{A}^* = \bigcup_{\alpha<\kappa} \mathscr{A}_\alpha$. ■

We now use the claim to recursively construct an elementary chain $\{\mathscr{A}_n : n < \omega\}$ such that $\mathscr{A}_{n+1} = \mathscr{A}_n^*$. Finally the model $\mathscr{B} = \bigcup_{n<\omega} \mathscr{A}_n$ is saturated: Let $F \subseteq B$ be a finite set $F \subseteq B$ and let $p \in S(F, \mathscr{B})$ be a complete type over B. Then there is $n < \omega$ such that $F \subseteq A_n$. By the above lemma $p \in S(F, \mathscr{A}_n)$ and, by construction, p is realized in A_{n+1} so, a fortiori, also in $\mathscr{B}$. □

The same proof as above gives the following theorem:

Theorem (5.4.9)**.** *If $\lambda > \aleph_0$ is inaccessible, then any T has a saturated model of size λ.*

4.4. **Atomic Models.**

Definition. A model is *atomic* (over X) if it realizes only isolated types (over X).

Constructing atomic models is not as easy (nor do they always exist) as constructing saturated models. We still have a counterpart to 4.3, however, since once a type is realized, it remains realized in every larger model, the counterpart is a bit weaker:

4.5. **Theorem** (Omitting types,M5.6.2,H6.2.1)**.** *Let T be a complete theory and p a non-isolated type. Then there is a countable model $\mathscr{A}$ omitting p, i.e. a model which does not realize p.*

We shall now aim to show that countable *atomic* models are initially universal, i.e. that they embed into all elementarily equvivalent models. Such models are called prime:

Definition. A model $\mathscr{A}$ is *prime* (over $X \subseteq A$) if it embeds into any model of $Th(\mathscr{A})$ (with an embedding fixing X pointwise).

Example. The model $\mathbb{Q}$ of a dense linear ordering without ends is prime.

Note. An infinite *prime* model is necessarily countable since $Th(A)$ has a countable model by the Löwenheim-Skolem theorem. Moreover, by theorem 4.5, it must be atomic. The following theorem shows that the converse is also true.

Theorem. *A countable atomic model is prime.*

Proof. Let $\mathscr{A}$ be a countable atomic model and let $\mathscr{B} \equiv \mathscr{A}$ be an infinite elementarily equivalent model. Enumerate A as $\{a_n : n < \omega\}$. We shall build a sequence $\langle f_n : n < \omega \rangle$ of elementary functions into $\mathscr{B}$ such that for each $n < \omega$

(1) $a_n \in dom(f_{n+1})$; and
(2) $f_n \subseteq f_{n+1}$.

Then $f \bigcup_{n<\omega} f_n$ will be an elementary embedding of $\mathscr{A}$ into $\mathscr{B}$. Let $f_0 = \emptyset$. Suppose now that we have constructed f_n. Since the type of a_n is isolated by atomicity, also the type of a_n over $\{a_0, \ldots, a_{n-1}$ is isolated (finitely many parameters can't change an isolated type into a nonisolated one). It follows that the image of this type via f_n is also isolated and hence realized in $\mathscr{B}$ by some $b_n \in B$. We can thus let $f_{n+1} = f_n \cup \{\langle a_n, b_n \rangle\}$. □

4.5.1. *TODO.*

Theorem (Existence,M6.4.14)**.** *The following are equivalent:*

(1) *has a prime model*
(2) *isolated types are dense in S*

Theorem (Ressayre,M6.4.11)**.**

5. Independent sets

Definition. A *dependence relation* $\preceq$ on M is a relation between elements of M and subsets of M satisfying:

(1) If $a \preceq X$ then $a \preceq X$
(2) If $a \preceq X$ then there is a finite $X_0 \subseteq X$ such that $a \preceq X_0$
(3) If $a \in X$ and $X \preceq Y$ (i.e. $x \preceq Y$ for each $x \in X$) then $a \preceq Y$
(4) If $a \preceq X \cup \{b\}$ and $a \not\preceq X$ then $b \preceq X \cup \{a\}$.

A set X is $\preceq$-*independent* if for each $x \in X$ we have $x \not\preceq X \setminus \{x\}$. A set $X \subseteq Y$ is a $\preceq$-*basis* for Y if it is $\preceq$-independent and $y \preceq X$ for every $y \in Y$. A set $X \subseteq M$ is $\preceq$-*closed* if for each $y \in M$ when $y \preceq X$ then $y \in X$.

Proposition. *Let $\preceq$ be a dependence relation on M. Then any independent subset of X can be extended to a basis of X and every two bases of X have the same cardinality.*

Proof. The first part is by Zorn's lemma. To show the second part, assume A,B are two bases of X and, aiming towards a contradiction, $|B| < |A|$. Assume first that one of the bases is infinite. Then for each $a \in A$ use 2. and the fact that B is a basis of X to find a minimal finite $B_a \subseteq B$ such that $a \preceq B_a$. Since B or A is infinite, $|[B]^{<\omega}| < |A|$ so there must be $a \in A$ such that $|\{a' \in A : B_{a'} = B_a\}|$ is infinite. Let $B_a = \{b_0,\dots,b_{n-1}\}$ and fix distinct $a_0,\dots,a_n$ with $B_{a_i} = B_a$. Since $a_0 \preceq \{b_1,\dots,b_{n-1}\} \cup \{b_0\}$ and, by minimality, $a_0 \not\preceq \{b_1,\dots,b_{n-1}\}$ we can use 4 to deduce $b_0 \preceq \{a_0,b_1,\dots,b_{n-1}\}$. Repeatedly applying 4 we deduce that $A_b \preceq \{a_0,\dots,b_{n-1}\}$. Then $b_n \preceq A_b$ contradicts the fact that B was independent. The case when $|B| < |A| < \omega$ is left as an exercise to the reader. □

Definition (M5.9.1)**.** An element $a \in \mathscr{A}$ is *algebraic* over $X \subseteq A$ (written as $a \preceq_{\mathscr{A}} X$) if there is an $L(X)$ formula φ such that $\varphi(\mathscr{A})$ is finite and contains a. We write $acl_{\mathscr{A}}(X)$ for the set of all elements algebraic over X. A set X is *algebraically closed* if $acl_{\mathscr{A}}(X) = X$.

Observation. *If $\mathscr{A} \equiv \mathscr{B}$ and $X \subseteq A \cap B$ is such, that inclusion is an elementary function from $A \cap B$ to both A and B then $y \preceq_{\mathscr{A}} X \iff y \preceq_{\mathscr{B}} X$ for any $y \in A \cap B$.*

Note. The above observation shows that algebraicity only depends on the theory $Th(\mathscr{A})$, not on the particular structure $\mathscr{A}$. This often allows us to drop the index from $\preceq_{\mathscr{A}}$ without danger of confusion.

We now aim to show that, although it fails in general, in some cases, $\preceq_{\mathscr{A}}$ is actually a dependence relation.

Exercise. Show that $\preceq_{\mathscr{A}}$ satisfies conditions 1.–3. of the definition of a dependence relation.

Example. Consider the structure $\mathscr{A}$ for $\mathscr{L} = \{\pi\}$ whose universe is ω^0 and $\pi^{\mathscr{A}}$ is the projection on the x-axis. Then $(0,0) \not\preceq_{\mathscr{A}} \emptyset$, $(0,0) \preceq \emptyset \cup \{(0,1)\}$ but $(0,1) \not\preceq \emptyset \cup \{(0,0)\}$. In particular $\preceq_{\mathscr{A}}$ does not satisfy condition 4. and is not a dependence relation.

Definition (M2.7.1)**.** A definable subset X of $\mathscr{A}$ is *minimal* if it is infinite but for every $L(A)$ definable set Y either $X \cap Y$ or $X \setminus Y$ is finite. It is *stronly minimal* if $X = \varphi(\mathscr{A})$ and $\varphi(\mathscr{B})$ is minimal for any elementary extension $\mathscr{B}$ of $\mathscr{A}$. A model $\mathscr{A}$ is minimal/strongly minimal if its universe A is. We say that a formula φ is *strongly minimal for $\mathscr{A}$* if $\varphi(\mathscr{B})$ is minimal in $\mathscr{B}$ for every elementary extension $\mathscr{B}$ of $\mathscr{A}$. A theory is *strongly minimal* if every model of T is minimal.

Theorem. *The algebraic dependence of a strongly minimal theory is a dependence relation.*

Proof. □

5.1. **Theorem** (H9.2.5)**.** *If $X \subseteq \mathscr{A}, Y \subseteq \mathscr{B}$ then any elementary bijection from X to Y can be extended to an elementary bijection between $acl_{\mathscr{A}}(X)$ and $acl_{\mathscr{B}}(Y)$.*

5.2. **Theorem** (H9.2.6)**.** *Suppose $\mathscr{A} \equiv \mathscr{B}$ and that C,D are independent minimal sets in $\mathscr{A}$ and $\mathscr{B}$ respectively. Then any injective function from C to D is elementary.*

5.3. **Corollary.** *Complete, strongly minimal theories are categorical.*

Proof. Let T be a complete strongly minimal theory and let $\mathscr{A},\mathscr{B}$ be two models of T of the same cardinality. Let X be a basis of $\mathscr{A}$ and Y a basis of $\mathscr{B}$. Since the algebraic closure of an infinite set X has the same cardinality as X we have $|X| = |acl(X)| = |A| = |B| = |acl(Y)| = |Y|$. In particular, there must be a bijection f between X and Y. Since T is strongly minimal, both X and Y are minimal and so, by 5.2, f is elementary. By 5.1 f can be extended to an isomorphism of $acl(X) = A$ and $acl(Y) = B$. □

In particular, Morley's theorem holds for strongly minimal theories. To prove it in general, we need to investigate how far a theory is from being strongly minimal. To this end we introduce the notion of a Morley rank.

6. Morley Rank

Definition (M5.7.1,M5.7.2). Let X be a $\mathscr{L}(A)$ definable subset of A^n and α an ordinal. We recursively define the *Morley rank* $RM_{\mathscr{A}}(X)$ of X as follows:

- $RM_{\mathscr{A}}(\emptyset) = -1$, $RM_{\mathscr{A}}(X) = 0$ if X is finite;
- if α is limit > 0 then $RM_{\mathscr{A}}(X) \geq \alpha$ if $RM_{\mathscr{A}}(\alpha) \geq \beta$ for all $\beta < \alpha$; and
- if $\alpha = \beta + 1$ then $R_M\mathscr{A}(X) \geq \alpha$ if there are infinitely many pairwise disjoint definable subsets $\{X_i : i < \omega\}$ of X such that $RM_{\mathscr{A}}(X_i) \geq \beta$ for each $i < \omega$.

Finally we say that $RM_{\mathscr{A}}(X) = \alpha$ if $RM_{\mathscr{A}}(X) \geq \alpha$ but not $RM_{\mathscr{A}}(X) \geq \alpha + 1$ and $RM_{\mathscr{A}}(X) = \infty$ if $RM_{\mathscr{A}}(X) \geq \alpha$ for all ordinals α.

Proposition (M5.7.4). *Let X, Y be two $\mathscr{L}(A)$ definable subsets of A^n. Then*

- *If $X \subseteq Y$ then $RM_{\mathscr{A}}(X) \leq RM_{\mathscr{A}}(Y)$*
- *If f is an automorphism of $\mathscr{A}$ then $RM_{\mathscr{A}}(X) = RM_{\mathscr{A}}(f[X])$*
- *$RM_{\mathscr{A}}(X \cup Y) = \max\{RM_{\mathscr{A}}(X), RM_{\mathscr{A}}(Y)\}$*

6.1. **Proposition** (M5.7.5). *If X is an $\mathscr{L}(A)$ definable subset of A^n of Morley rank α then there is a maximal integer d such that X can be written as a disjoint union of d sets of Morley rank α. This integer is called the Morley degree of X and denoted by $GM_{\mathscr{A}}(X)$.*

It might be tempting to assume that Morley's rank of a set $X \subseteq A$ only depends on the complete theory $Th(\mathscr{A})$. However, this is not the case as, in general, it will also depend on the ambient structure $\mathscr{A}$. However, as the following proposition shows, if the structures are sufficiently saturated, then the rank indeed only depends on the theory.

6.2. **Proposition** (M5.7.9). *If $\mathscr{A}$ and $\mathscr{B}$ are two $\aleph_0$-saturated models of a complete theory T and $X \subseteq A, B$ is both A-definable and B-definable. Then $RM_{\mathscr{A}}(X) = RM_{\mathscr{B}}(X)$.*

(without proof)

Remark. According to the previous proposition (6.2), we may define the Morley Rank of a set X without refering to the model $\mathscr{A}$ by saying that the rank is the rank in any $\aleph_0$-saturated model where X is definable. Similarly we may define the *Morley Rank* of a complete theory to be the Morley Rank of the set M (definable by $x = x$) for any $\aleph_0$-saturated model $\mathscr{M}$.

Definition. A complete theory is *totally transcendental* if its Morley rank is $< \infty$.

We will show that categorical theories are *totally transcendental* and that transcendental theories are stable which are sufficiently close to being minimal that they also satisfy Corollary 5.3.

6.3. **Theorem.** *A complete theory is* not *totally transcendental iff there is an $\aleph_0$-saturated model of T and a tree of definable sets $\{X_s : s \in 2^{<\omega}\}$ such that $X_s \subseteq X_t$ whenever $s \subseteq t$, and X_{s0}, X_{s1} is a partition of X_s for each $s \in 2^{<\omega}$.*

Proof. Let $\mathscr{A}$ be any $\aleph_0$-saturated model of T. If T is not totally transcendental, then $RM_{\mathscr{A}}(A) = \infty$. Since any set of rank ∞ can be split into two sets of rank ∞ (to see this note that there must be a maximal α such that every definable subset has either rank α or ∞) it is easy to build the tree by induction. On the other hand suppose that $\{X_s : s \in 2^{<\omega}\}$ is such a tree. We show, by induction on $\beta \in On$, that each X_s has rank $\geq \beta$. For $\beta = -1$ this is true and for β limit this is trivial. For $beta = \gamma + 1$, we know by the inductive assumption that $RM_{\mathscr{A}}(X_s) \geq \gamma$ for each $s \in 2^{<\omega}$. Then for each $\{X_{s0^n1} : n < \omega\}$ is a partition of X_s into infinitely many sets of rank $\geq \gamma$ so $RM_{\mathscr{A}}(X_s) \geq \gamma + 1$. □

7. Stability

Definition (H9.4). A complete theory T is *unstable* if there is a formula $\varphi(\bar{x}, \bar{y})$ and a model $\mathscr{A}$ of T containing tuples of elements $\{\bar{a}_n : n < \omega\}$ such that for all $n, m < \omega$

$$\mathscr{A} \models \varphi(\bar{a}_n, \bar{a}_m) \quad \text{iff} \quad n < m$$

If a theory is not unstable, it is *stable*.

Although it is not immediately clear from the definition, stable theories are theories which do not have many types:

Definition (H9.4)**.** A complete theory is *λ-stable* if for every model $\mathscr{A}$ of T and $X \subseteq A$ of size at most λ there are at most λ-many complete 1-types over X (i.e. $|S_1(\mathscr{A}, X)| \leq \lambda$)

We now aim to show that stability is equivalent to λ-stability. The following is a technical lemma, showing that we could as well have considered n-types in the definition of λ-stability.

Lemma. *If $\mathscr{A}$ is model of a λ-stable theory and $X \subseteq A$ is of size at most λ then $|S_n(\mathscr{A}, X)| \leq \lambda$ for every $n < \omega$.*

Proof. The proof is by induction on n. Assume $\mathscr{A}$ is a λ-stable theory. The case $n = 1$ follows from the definition. So assume we have proved it for n. We will show that it holds for $n+1$. First notice that if π is a projection from $X^{n+1} \to X^i$ then the image via π of any complete $(n+1)$-type is a complete i-type. Aiming towards a contradiction assume that $|S_{n+1}(X, \mathscr{A})| \geq \lambda^+$. By the inductive assumption there are at most λ-many distinct projections onto the first n-coordinates, so there must be λ^+-many distinct projections onto the last coordinates. But this contradicts λ-stability. $\square$

Exercise (optional)**.** If T is a complete strongly minimal theory in a countable language then it is ω-stable.

Proof. (Hint) Every model of T has only a single non-algebraic type. $\square$

Theorem (H9.4.2)**.** *A complete theory T is stable iff it is λ-stable for some infinite λ.*

Proof of the 'if' part. Assume that T is not stable as witnessed by a formula φ with n free parameters. We shall show that it is not λ-stable for any infinite λ: fix λ and let $\leq$ be a linear order on $R \supseteq P$ such that P is dense in R and $|P| \leq \lambda < |R|$ (e.g., if μ is the least cardinal such that $\lambda < 2^\mu$ then $2^{<\mu}$ is dense in 2^μ in the lexicographic ordering). Expand the language of T by adding a new n-tuple of constants c_p for each $p \in P$. Then, by compactness, the theory $T \cup \{\varphi(c_p, c_q) : p \leq q \in P\}$ has a model $\mathscr{A}$. Its reduct to the original language is a model of T and this model has a distinct complete n-type for each $r \in R$ — $\{\varphi(c_p, x) : p \leq r\} \cup \{\varphi(x, c_p) : r \leq p\}$. $\square$

The 'only-if' part of the above theorem is more complicated and we will not need it. The interested reader is refered to Hodges's book.

Theorem (H9.4.5)**.** *If T is a complete totally transcendental theory, then T is λ-stable for all $|\mathscr{L}| \leq \lambda$ (and hence stable).*

Proof. Let T be a complete totally transcendental theory T and, aiming towards a contradiction, assume it is not λ-stable for some $\lambda \geq |\mathscr{L}|$, i.e. $\lambda^+ \leq |S_1(X, \mathscr{A})|$ for some model $\mathscr{A}$ and $X \in [A]^\lambda$. Without loss of generality we may assume that $\mathscr{A}$ is $\aleph_0$-saturated and that $\mathscr{L} = \mathscr{L}(X)$. Let $\mathcal{V} = \{B : |\hat{B}| \geq \lambda^+\}$. Let $S = S_1(X, \mathscr{A}) \setminus \bigcup \{\hat{B} : |\hat{B}| < \lambda^+\}$. Then $|S| \geq \lambda^+$. Now we recursively build a tree as in Theorem 6.3 which shows that T is not totally transcendental. This can be done since if $B \in \mathcal{V}$ then it is contained in at least two distinct $p, q \in S$. Since p, q are distinct, there must be $C \in p \setminus q$. Then $B \cap C \in \mathcal{V}$ and $B \setminus C \in \mathcal{V}$, so S can be split into two elements in $\mathcal{V}$. $\square$

8. Morley's Theorem

Definition. A theory T is κ-categorical if it has, up to isomorphism, only a single model of size κ.

For the rest of this section assume that T is a κ-categorical theory (in a *countable* language) for some uncountable κ.

We first aim to show that T is simple, in the sense of Morley's rank, i.e. totally transcendental. The proof relies on Ehrenfeucht-Mostowski models which we did not cover. The following definition and theorem state what is needed for our purposes:

Definition. Let $(X,\leq)$ be a linear order, let Γ be a family of formulas of a language $\mathscr{L}$ and let $\mathscr{A}$ be an $\mathscr{L}$-structure with $X \subseteq A$. Then we say that $(X,\leq)$ is an Γ-*indiscernible* sequence, if for each $\varphi(x_0,\dots,x_n) \in \Gamma$ and each two increasing $(n+1)$-tuples $a_0 < \cdots < a_n$ and $b_0 < \cdots < c_n$ we have

$$\mathscr{A} \models \varphi(a_0,\dots,a_n) \iff \mathscr{A} \models \varphi(b_0,\dots,b_n).$$

A sequence is *indiscernible* if it is $Fml(\mathscr{L})$-indiscernible.

The following is a special case of the Ehrenfeucht-Mostowski theorem. We state it here without proof.

8.1. **Theorem** (H9.1.5, H9.1.6)**.** *If $\mathscr{A}$ is a structure such that $Th(\mathscr{A})$ is a Skolem Theory and μ is any ordinal then there is a model $\mathscr{E}M(\mathscr{A},\mu)$ of $Th(\mathscr{A})$ such that $\mu \subseteq EM(\mathscr{A},\mu)$, μ generates $EM(\mathscr{A},\mu)$ and mu is an indiscernible sequence in B.*

8.2. **Proposition** (H9.1.7b)**.** *If $X \subseteq EM(\mathscr{A},\mu)$ then there are at most $|\mathscr{L}|+|X|$-many complete 1-types over X realized in $EM(\mathscr{A},\mu)$.*

Proof. Since μ generates $EM(\mathscr{A},\mu)$, for each $a \in X$ there is a term t_a and a tuple $\overline{b_a} \in [\mu]^{<\omega}$ such that

$$a = t_a^{EM(\mathscr{A},\mu)}(\overline{b}).$$

Let $W = \bigcup\{b_a : a \in X\}$. If $s(\overline{x})$ is a term and $c \in [\mu]^{<\omega}$, then, by indiscernibility of μ, the complete type of $s^{EM(\mathscr{A},\mu)}(\overline{c})$ over X is completely determined by the positions of the elements of $\overline{c}$ relative to W, i.e. by the set $\{c^{+W} : c \in \overline{c}\} \in [W]^{<\omega}$, where $c^{+W} = \min W \setminus c$. It follows that the value of the term $s(\overline{x})$ in $EM(\mathscr{A},\mu)$ can realize at most $|W|^{<\omega} = |W|+\omega$-many types. Since $EM(\mathscr{A},\mu)$ is generated by μ and since there are at most $|L|$-many terms the result follows. □

We can now show that a categorical theory is totally transcendental:

Theorem (H9.5.2)**.** *T is totally transcendental.*

Proof. We first show that T is stable. By Skolemizing, we can assume without loss of generality that T is a Skolem theory. By 8.1 we can find a model B of T generated by κ as an indiscernible sequence. We show that T is stable. Otherwise there is a model $\mathscr{A}$ of T and a countable $Y \subseteq A$ with $|S_1(\mathscr{A},Y)| > \omega$. By Löwenheim-Skolem we can find a $\mathscr{A}'$ of size κ containing Y. Then $|S_1(\mathscr{A}',Y)| > \omega$ but, by categoricity $A' \simeq B$ and, by 8.2, $|S_1(B,Y)| \leq \omega$ — a contradiction.

It remains to show that T is totally transcendental. Aiming towards a contradiction assume it is not. By 6.3, there is a "partition" tree $\{X_s : s \in 2^{<\omega}$ of definable sets. Each branch through this partition generates a distinct complete type contradicting the stability of T. □

Theorem. *T has a countable prime model $\mathscr{A}_0$.*

(without proof)

Theorem (H9.5.6)**.** *T has a strongly minimal formula φ with parameters from $\mathscr{A}_0$.*

(without proof)

Definition. A formula φ is *one-cardinal* for a theory T, provided $|A| = |\varphi(A)|$ for each model A of T.

Proposition. *The strongly minimal formula for T is one-cardinal.*

(without proof)

8.3. **Theorem** (Morley's theorem,H9.5.11)**.** *If $\mathscr{A}$ is a model of T then it can be constructed as the unique prime model over $\varphi(A)$. In particular, $|A| = |\varphi(A)|$. (without proof)*

Theorem (Morley's theorem)**.** *If T is κ-categorical for some uncountable κ then it is λ-categorical for every uncountable λ.*

Proof. Suppose $\mathscr{A},B$ are two models of T of the same cardinality and assume, without loss of generality, that $\mathscr{A}_0 \subseteq \mathscr{A},B$.

- By the previous theorem $\varphi(\mathscr{A})$ and $\varphi(\mathscr{B})$ have the same cardinality
- Let $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$ be a maximal independent subsets of $\varphi(\mathscr{A})$ and $\varphi(\mathscr{B})$ respectively.
- Since the language is countable, $|\varphi(\mathscr{A})| = |A| = |B| = |\varphi(\mathscr{B})| = \lambda$ and since every element of $\varphi(\mathscr{A})$ and $\varphi(\mathscr{B})$ is algebraic over $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$ respectively, $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$ have the same cardinality
- Let g be a bijection between $I_{\mathscr{A}}$ and $I_{\mathscr{B}}$, by 5.2 g is elementary so by 5.1 it can be extended to an elementary function h from $\varphi(\mathscr{A})$ to $\varphi(\mathscr{B})$.
- We may assume that we work in some large saturated model $\mathscr{M}$. By saturation, h can be extended to an automorphism f of $\mathscr{M}$. Then by 8.3 B is isomorphic to B. Since A is isomorphic to $f(A)$ we are done.

$\square$

References

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