Relevant Epistemic Logic

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Abstract

We propose a system of epistemic logic much weaker than the standard modal frameworks, which is based on the relevant logic R, extended with a distinctive epistemic modality K. The intended interpretation is that $K\varphi$ holds (relative to a given information state s) if there is another information state (a source) available at s, confirming φ.

1 Introduction

The problem of representation of epistemic states of and their changes has been discussed for a long time. The classical solution takes knowledge operator as a standard necessity-like modal operator and interprets the standard modal axioms (K, T, 4, 5) as epistemic properties (closure, truth, positive introspection, negative introspection). The most popular formalization (used also in computer science) is based on the epistemic version of S5, in which knowledge turns out to be an indistinguishability between epistemic states.

This approach has been extensively criticized (see, e.g., Fagin et al. (2003) and Duc (2001)) for being unrealistically strong. Agents it represents are 'too perfect'—they are, e.g., logically omniscient (they know all the logical truths) and fully introspective (they are explicitly aware of their both positive and negative knowledge). For these reasons such representations are sometimes called epistemic logics of *implicit* knowledge.

The introspection axioms can be omitted if we use systems weaker than S4, however, the omniscience appears in all systems of normal epistemic logics. One possibility to solve this was to use dynamic epistemic logic. In Duc (2001) we can find solutions based on modifications of standard Kripke semantics (awareness and impossible worlds) as well as solutions based on a combination of temporal and epistemic logic and complexity approaches (algorithmic knowledge). In our approach we shall avoid omniscience by using a weaker system than that of a normal modal logic, namely the framework of distributive relevant logic.

There have been some proposals combining epistemic and relevant frameworks (see Cheng (2000) and Wansing (2002)), but as they have a different aim then our approach, we are not going to discuss them here. From a purely technical point of view there are a number of ways to introduce modalities in the relevant framework—Greg Restall in Restall (2000) provides a nice general overview. We take a different approach—instead of adding modalities externally we use notions already contained in the relevant framework to build our knowledge operator. There are several reasons for that. The relevant framework is complex on its own and adding completely independent modalities on the top of it would make it even harder to deal with. Second reason is an interpretation of the relevant framework itself. It has been criticized for its seeming (in our opinion) non-intuitivness and a lack of a generally accepted clear philosophical interpretation. Providing an epistemic reading to some of the components of the relevant logic we would like to contribute to the collection of its interpretations. We also think that relevant framework very naturally represents our prototypical example of a rational agent—a scientist dealing with scientific data.

2 Agent

Imagine a a scientist performing some experiments in a laboratory. Besides data from her own observations she has obviously access to other sources of data relevant for her research (articles, databases, etc.). Information available to her has two basic ingredients: experimental data ('facts')—inputs and outputs of experiments/observations and 'laws'—generalizations extracted from the experimental data.

If we consider these two kinds of data from the point of view of a logical framework, we can, with some simplification, say that basic 'facts' are typically represented by atoms and their conjunctions and disjunctions, while basic 'laws' are represented by conditionals (and their combinations).

Abstracting from our current motivation, the question of an adequate interpretation of conditionals has a long history. It has been discussed since the beginning of 20th century (Hugh MacColl, Carl Irvin Lewis). The majority of solutions which have been produced agree, that a material implication does not reflect intuitions about conditionals and the way they are used in standard communication and that an adequate representation should require some connection between the antecedent and the consequent. This is in an accordance with our 'laboratory' motivation. The connection we are looking for is a regularity or a law-like connection between antecedent and consequent data and it is clear that the material implication is not an appropriate representation of this kind of connection as, among other things, it connects any two arbitrary formulas. For example, it holds for any α, β that

1. $\alpha \rightarrow (\beta \rightarrow \alpha)$
2. $(\alpha \wedge \neg\alpha) \rightarrow \beta$
3. $\alpha \rightarrow (\beta \vee \neg\beta)$

In our epistemic reading the material implication would make a 'law' from every two 'facts', which would obviously make the representation useless. It also admits not very useful 'laws' with the tautological consequent (as in 3).

The list of undesired properties does not end here. Our prototypical scientist is obviously faulty—due to an error her data may contain some contradicting pieces. But material implication obviously cannot deal with errors in the data. One such error corrupts all the remaining data (from a contradiction everything follows, see 2).

The tautologies 1–3 are just examples of the paradoxes of material implication. The fact, that these 'paradoxes' were completely solved only in the systems of relevant logics, the obvious choice for a conditional for our scientific agent is relevant implication.

3 Relational semantics

We criticized the modal tradition of accessing epistemic logic, however, like to keep some of its basic principles. We would like to have a notion closed to that of a possible world in the sense of a set of formulas representing an epistemic state of an agent. We also want to make agent's knowledge dependent not only on her current epistemic state, but also on some related or alternative epistemic states. These requirements naturally lead to some sort of a relational semantics. From what we discussed above it, should be clear that we cannot use a standard Kripke semantics with possible worlds and a binary accessibility relation, we need a more general relational structure.

Our agent is imperfect (in fact all the human agents are). One consequence of it we already mentioned: she can obtain contradicting data. She also might be unable to decide about truth/falsity of every formula (of a given language). Thus we have to weaken the notion of possible world in order to account for these cases. We shall call the new entity allowing to accommodate an inconsistent and/or incomplete data *situation*. We also replace the standard relation of epistemic accessibility with the relation of *independent confirmation*. This notion will be introduced in the section 4.

Our point of departure will be the distributive relevant logic R of Anderson and Belnap. Although the most natural way to introduce relevant logics is certainly proof theoretical (see, e.g., Paoli (2002)), we base our framework on the relational Routley-Meyer semantics, as developed by Mares (2004), Restall (1999), Paoli (2002), and others, on which we shall define epistemic modalities.

We give an informal exposition of structures in the relevant frame and definition of connectives (for formal definitions see the Appendix A).

3.1 Relevant frame

A relevant frame is a structure $\mathbf{F} = \langle S, L, C, \trianglelefteq, R \rangle$, where S is a non-empty set of situations (states), $L \subseteq S$ is a non-empty set of designated *logical situations*, $C \subseteq S^2$ is a *compatibility* relation, $\trianglelefteq \subseteq S^2$ is a relation of *involvement*, $R \subseteq S^3$ is an *relevance* relation.

A model $\mathbf{M}$ is a relevant frame with the relation $\Vdash$, where $s \Vdash \varphi$ has the same meaning as in Kripke frames—that s carries the information that the formula φ is true ($\varphi \in s$ if we consider states to be sets of formulas).

Situations Situations (sometimes also called information states) play in our framework the same role as possible worlds in Kripke frames. We assume, they consist of data immediately available to the agent. Like possible worlds, we can see situations as sets of formulas, but, unlike possible worlds, situations might be incomplete (neither φ nor $\neg\varphi$ is true in s) or inconsistent (both φ and $\neg\varphi$ are true in s).

Conjunction and disjunction Classical (weak) conjunction and disjunction correspond to the situation when the agent combines local data, i.e., data from her current situation. They behave in the same way as in the case of classical Kripke frames—their validity is given locally:

$$s \Vdash \psi \wedge \varphi \text{ iff } s \Vdash \psi \text{ and } s \Vdash \varphi$$

$$s \Vdash \psi \vee \varphi \text{ iff } s \Vdash \psi \text{ or } s \Vdash \varphi$$

Weak connectives are the only ones which are defined locally. The truth of negation and implication depends also on the data in situations, related to the actual ones, so they are modal by nature. It is possible to define strong conjunction and disjunction as well (see the Appendix A).

Implication Implication is a modal connective in the sense that it depends on a neighborhood of a current situation, which is given by the ternary *relevance* relation R. In fact it is analogous to the strong (necessary) implication in a standard Kripke frame. We know that an implication $(\varphi \to \psi)$ is necessarily true in a given world in a Kripke frame iff in all worlds, accessible from the given one, where the antecedent holds, the consequent holds as well. In other words, the implication $(\varphi \to \psi)$ holds necessarily if it holds through all the neighborhood of the given world. We can read the relevant implication in a very same way, except the neighborhood of a situation s is given by pairs of situations (y, z) such that (s, y, z) are related by R. We shall call y, z antecedent and consequent situations, respectively. We say that the implication $(\varphi \to \psi)$ holds at the situation s iff it is the case that for every antecedent situation y where φ (the antecedent of the implication) holds, ψ (the consequent of the implication) holds at the corresponding consequent situation z.

$$s \Vdash (\varphi \to \psi) \text{ iff } (\forall y, z \in S)(Rsyz \text{ implies } (y \Vdash \varphi \text{ implies } z \Vdash \psi))$$

The relation R reflects in our interpretation actual experimental setups. Antecedent situations correspond to some initial data (outcome of measurements or observations) of some experiment, while the related consequent situations correspond to the corresponding resulting data of the experiment. Implication then corresponds to some (simple) kind of a rule: if I observe in my current situation, that at every experiment (represented by a couple antecedent–consequent situation) each observation of φ is followed by an observation of ψ, then I accept 'ψ follows φ' as a rule.

Negation In Kripke models the *negation* of a formula φ is true at a world iff φ is not true there. As situations can be incomplete and/or inconsistent, this is not an option any more. Negation becomes a modal connective and its meaning depends on the worlds related to the given world by a binary modal relation C known as *compatibility*. Informally we can see the compatible situations as information sources our scientist wants to be consistent with. (Imagine data of different research groups working on related subject.) Relevant negation does not correspond straightforwardly to 'necessary false'. We do not require that the negated formula in question is false in the neighborhood of the given world, we just require no world in the neighborhood contains this formula unnegated.

The formula $\neg\varphi$ holds at $s \in S$ iff it is not 'possible' (in the standard modal sense with respect to the relation C) that φ; at no situation s', compatible with ('accessible from') the situation s, it is the case that φ (either s' is incomplete with respect to φ or $\neg\varphi$ holds there).

$$s \Vdash \neg\varphi \text{ iff } (\forall s' \in S)(sCs' \text{ implies } s' \nVdash \varphi)$$

Informally speaking, the agent can explicitly deny some information (a piece of data) only if no research group in her neighborhood claims it is true. This condition also has a normative side: she has to be skeptical in the sense that she denies everything not positively supported by any of her colleagues (in the situations related to her actual situation).

If we want to grant negative facts the same basic level as positive facts, we can read the clause for the definition of compatibility in the other direction: the agent can relate her actual situation just to the situations which do not contradict her negative facts.

Properties of the compatibility relation obviously determin the kinds of negation obtained. We shall not discuss them here, let us just note, that we assume compatibility is symmetric, but it is in general neither reflexive (inconsistent situations are not self-compatible) nor transitive. (For a formal definition see the Appendix A.)

Logical situations The framework we presented so far is very weak: there are just few tautologies valid in all situations and some of the important ones—those being usually considered as basic logical laws—are missing. For example the almost uniformly accepted identity axiom $(\alpha \rightarrow \alpha)$ and the Modus Ponens rule fail to hold in every situation.

This is closely connected to the question how to define truth in a relevant frame (model). If we take a hint from Kripke frames, we should equate truth in a frame with truth in every situation. But this would gives us an extremely weak system with some very unpleasant properties (cf. Restall (1999)). Designers of relevant logics took a different route; instead of requiring truth in all situations, they identify the truth in a frame just with the truth in all logically 'well behaved' situations. These situations are called *logical*. In order to satisfy the 'good behavior' of a situation l it is enough to require that all the information in any antecedent situation related to l is contained in the corresponding consequent situation as well: for each $x, y \in S$, $Rlxy$ implies $|x| \subseteq |y|$, where $|s|$ is the set of all formulas, which are true in the situation s.

It is easy to see that situations constrained in this way validate both the identity axiom and (implicative) Modus Ponens.

Involvement Involvement is a relation resembling the persistence relation in intuitionistic logic—we can see it as a relation of information growth. However not every two situations which are in inclusion with respect to the validated formulas are in the involvement relation. We require that such an inclusion is observed or witnessed and only the logical situations can play the role of a witness.

$$x \trianglelefteq y \text{ iff } (\exists l \in L)(Rlxy)$$

This completes our exposition of relational semantics for relevant logics. We now move to epistemic modalities.

4 Knowledge

As we already mentioned, we are not going to introduce epistemic modalities as an external notion. The relevant framework with the motivation we presented already contains enough modal notions to define an epistemic operator we need. We therefore decided to use these notions rather than introduce new ones.

In the classical epistemic frame what an agent knows in a world w is defined as what is true in all epistemic alternatives of w, which are given by the corresponding accessibility relation. Our idea of the agent as a scientist processing some kind of data requires a different approach.

We assume our agent in her current situation s observes (has a direct approach to) some data, represented by formulas which are true at s. She is aware of the fact that these data might be unreliable (or even inconsistent). In order to accept some of the current data as knowledge the agent requires a confirmation from some 'independent' sources.

We require from a source of a current situation s to satisfy the following conditions:

1. A source shall be more elementary (it should not contain more data) than the current situation. (A source is below s in the $\trianglelefteq$-relation.)
2. The data from the source should not contradict the data in the current situation. (A source is compatible with s.)
3. A source shall be different from the current situation.

Definition 4.1 (Knowledge)**.** $s \Vdash K\varphi$ iff

$$(\exists x \in S)(sC^{\triangleleft}x \text{ and } x \Vdash \varphi),$$

where

$$sC^{\triangleleft}x \text{ iff } sCx,\ x \trianglelefteq s, \text{ and } x \neq s.$$

In short, φ is known iff there is an source ('lower' compatible situation different from the actual one) validating φ.

We allowed our agent to deal with inconsistent data in order to get a more realistic picture. However, the agent should be able to separate inconsistent data. The modality we introduced provides us with just such an appropriate filter. Let us assume both φ and $\neg\varphi$ are in s (e.g., our agent might received such inconsistent information from two different sources). The agent considers both φ and $\neg\varphi$ as available data, but neither of them is confirmed information as according to the definition, no situation compatible to s can contain either φ or $\neg\varphi$.

4.1 Basic properties

It is to be expected that our system blocks all the undesirable properties of both material and strict implication. Moreover, we ruled out the validity of some of the properties of 'classical' epistemic logics that we have criticized, in particular, both positive and negative introspection, as well as some closure properties.

Let us have a relevant frame $\mathbf{F} = \langle S, L, C, \trianglelefteq, R\rangle$. Recall that the truth in the frame $\mathbf{F}$ corresponds to the truth in the logical situations of $\mathbf{F}$ (under any valuation). We will also consider a stronger notion—truth in all situations of $\mathbf{F}$

(under any valuation). This notion is interesting from the point of view of our motivation as our agent might happen to be in other actual situations than the logical ones.

Factivity Our approach makes the truth axiom **T** valid. For any situation (not only a logical one) $s \in S$, if φ is known at s ($s \Vdash K\varphi$), then there is a $\trianglelefteq$-lower compatible witness with φ true, which makes φ to be true at s as well. Thus, formula

$$K\alpha \to \alpha$$

is valid.

K-axiom In our interpretation the validity of the axiom **K** is not well motivated. **K** would in fact correspond to a 'distribution of confirmation': if an implication is confirmed, then the confirmation of the antecedent implies the confirmation of the consequent. Which does not need to be the case.

$$\not\models K(\alpha \to \beta) \to (K\alpha \to K\beta)$$

Introspection We defined knowledge as independently confirmed data. In this reading the axioms **4** and **5** rather than to introspection correspond to a 'second order confirmation' (if α is confirmed then the confirmation of α is confirmed as well, similarly for the negative introspection). It is easy to see that both axioms fail.

$$\not\models K\alpha \to KK\alpha$$

$$\not\models \neg K\alpha \to K\neg K\alpha$$

Necessity and negation There is a difference between $s \nVdash K\varphi$ and $s \Vdash \neg K\varphi$. The former simply says that φ is not confirmed at the current situation s, while the latter is stronger (at least in the case of selfcompatible situations), it says that φ is not confirmed in any situation compatible with s. From this point of view it is uncontroversial that both $K\varphi$ (confirmation in the current situation) and $\neg K\varphi$ (the lack of confirmation in the compatible situations) might be true in some situation s (in this case s is not compatible with itself).
On the other hand no information can be confirmed in a current situation, if the corresponding negative data are available

$$\not\models K(\varphi) \wedge \neg\varphi$$

4.2 Closure properties

In the introduction we criticized too strong closure properties of the standard modal representations of knowledge. In fact the question how strong conditions shall be imposed on epistemic states to obtain an adequate representation is one of the crucial choices of the knowledge representation. It is also closely related to the problem of logical omniscience.

We can see the machinery of 'logical expansion' as having two basic ingredients. One is knowledge of all the tautologies of the logical system in question guaranteed by the necessity rule. The other is Modal Modus Ponens, which produces all consequences of any new piece of (non-logical) information.

Our system turns out to be extremely weak and avoids both of these closure properties and some more. It can be seen as anti-logical and pragmatic - in a sense that our agent believes (accepts) just what is (or was) observed. Even the data corresponding to logical laws have to be confirmed.

Necessitation rule The necessitation rule,

$$\frac{\varphi}{K\varphi}$$

common to all normal epistemic logics, guarantees among other things that all the tautologies of the logical system in question are known. In our framework this would mean that all the logical truths are confirmed. This is in general not the case. Let us assume that φ is valid formula (i.e., $l \Vdash \varphi$, for every logical situation l). The necessity rule would imply the validity of $K\varphi$. However, for $l \Vdash K\varphi$ an confirmation from a different source is required, so there must be a situation x such that $x \Vdash \varphi$ and $lC^{\triangleleft}x$, which, in general, does not need to be the case.

Modal Modus Ponens Closure of knowledge with respect to logical consequence, which is a part of logical omniscience (if an agent knows both φ and $\varphi \rightarrow \psi$, then she knows ψ as well) is forced by the validity of the modal Modus Ponens:

$$\frac{K\alpha \quad K(\alpha \rightarrow \beta)}{K\beta}$$

It is easy to see that it does not not hold in our system. As we noted above, **K** is in fact a 'distribution of confirmation'. If both an implication and its antecedent are confirmed, there is no reason the consequent needs to be confirmed as well.

Let us note, that the weaker version of modal Modus Ponens holds

$$\frac{K\alpha \quad K(\alpha \to \beta)}{\beta}$$

however, it cannot be considered as any kind of omniscience. It just says that if both α and $(\alpha \to \beta)$ are confirmed, then β is a part of currently available data.

This rule holds not only in logical situations, but in all situations. If $K\alpha$ and $K(\alpha \to \beta)$ are true in an $s \in S$, then $s \Vdash \alpha$ and $s \Vdash \alpha \to \beta$ because of **T** axiom. It follows from the assumption $Rsss$ and the definition of implication, that $s \Vdash \beta$ as well.

Contradiction Contradiction in relevant logic is non-explosive: φ and $\neg\varphi$ might hold in a contradictory situation, but it does not entail an arbitrary formula ψ. (this would require an R-connection to situation where ψ holds).[1]

$$\not\models (\varphi \wedge \neg\varphi) \to \psi$$

As we noted above, a contradiction cannot be known (it is never confirmed).

$$\not\models K(\varphi \wedge \neg\varphi)$$

This has a trivial consequence, that knowledge of contradiction implies anything ($\models K(\varphi \wedge \neg\varphi) \to \psi$), so, in particular knowledge of contradiction implies knowledge of anything ($\models K(\varphi \wedge \neg\varphi) \to K(\psi)$). Nevertheless this does not lead to any kind of explosion as there is no such situation in which the antecedent is true. In standard models, $K(\varphi \wedge \neg\varphi)$ is never true either, but the reason is that $\varphi \wedge \neg\varphi$ is not true in any state (possible world). In our framework the situation is different: $\varphi \wedge \neg\varphi$ can be true in some situation (the agent obtained contradictory data), but $K(\varphi \wedge \neg\varphi)$ cannot.

Adjunction Modal adjunction also does not hold—if $K\alpha$ and $K\beta$ are true in s, then obviously $(\alpha \wedge \beta)$ is true there, but $K(\alpha \wedge \beta)$ need not be.[2] Our agent is really careful here. Even if each of α and β are confirmed separately, their conjunction is not accepted as knowledge, unless there is a single source confirming both of them (which in general does not need to be the case).

[1] The explosion does not occur even in the case of the strong conjunction; $(\varphi \otimes \neg\varphi) \to \psi$ does not hold.

[2] The same negative result holds also for strong conjunction. If $K\alpha$ and $K\beta$ are true in s, then $(\alpha \otimes \beta)$ is true in s (because of the truth axiom **T** and, moreover, $Rsss$ for this case), but $K(\alpha \otimes \beta)$ need not be true in s.

Modal disjunction rule In our system knowledge distributes with disjunction. It holds that

$$\frac{K(\alpha \vee \beta)}{K\alpha \vee K\beta}$$

Given a disjunction is confirmed in a current situation by a certain source, one of the disjuncts must be confirmed by it as well, because a disjunction is true at the source if at least one of the disjuncts is.

5 Conclusion

The original motivation for the project of relevant epistemic framework was an idea of a knowledge representation, which avoids the frequently criticized features of traditional modal representation, in particular, logical omniscience. We defined a new epistemic operator using standard parts of the relational semantics of distributive relevant logic—the relations of compatibility and involvement. The motivation we had in mind was an idea of a scientific agent-observer, whose knowledge is identified with the notion of independently confirmed data.

We obtained an epistemic operator which is extremely weak and has almost no closure properties. Our initial requirements were met, but we are aware of the fact, that our solution will not satisfy everybody. First problem is the use of relevant framework, which, for many, seems to be too complex and lacking a clear interpretation. We find on a contrary the framework very elegant and hope our motivation suggested one more interpretation of the the framework. Another problem is that the notion of knowledge represented by our operator is very weak—it certainly does not cover all the aspects of knowledge as it is generally understood and in particular it does not allow straightforwardly represent any reasoning processes. It was not our aim to define a generally applicable representation (and it does not seem there is one on the market) and the standard representations are rather 'overdetermined', so our attempt might be seen as showing the 'lower' end of the scale.

Our project is still work in progress. An earlier stage of this project was reflected in the article Majer and Peliš (2009). Another article dealing with the questions of axiomatization, completeness and properties of our system is in preparation.

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A Relevant logic R

There are more formal systems that can be called relevant logic. From the proof-theoretical viewpoint, all of them are considered to be substructural logics (see Restall (2000) and Paoli (2002)). Here we present the axiom system and (Routley-Meyer) semantics from Mares (2004) with some elements from Restall (1999).

A.1 Syntax

We use the language of classical propositional logic with signs for atomic formulas $\mathcal{P} = \{p, q, \ldots\}$, formulas being defined in the usual way:

$$\varphi ::= p \mid \neg\psi \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \psi_1 \rightarrow \psi_2$$

Axiom schemes

1. $A \rightarrow A$
2. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
3. $A \rightarrow ((A \rightarrow B) \rightarrow B)$
4. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
5. $(A \wedge B) \rightarrow A$
6. $(A \wedge B) \rightarrow B$
7. $A \rightarrow (A \vee B)$
8. $B \rightarrow (A \vee B)$
9. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$
10. $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$
11. $\neg\neg A \rightarrow A$

12. $(A \to \neg B) \to (B \to \neg A)$

Strong logical constants $\otimes$ (group conjunction, fusion) and $\oplus$ (group disjunction) are definable by implication and negation:

- $(A \oplus B) \equiv_{def} \neg(\neg A \to B)$
- $(A \otimes B) \equiv_{def} \neg(\neg A \oplus \neg B)$

Rules

Adjunction From A and B infer $A \wedge B$.

Modus Ponens From A and $A \to B$ infer B.

A.2 Routley-Meyer semantics

An R-frame is a quintuple $\mathbf{F} = \langle S, L, C, \trianglelefteq, R\rangle$, where S is a non-empty set of situations and $L \subseteq S$ is a non-empty set of logical situations. The relations $C \subseteq S^2$, $\trianglelefteq \subseteq S^2$, and $R \subseteq S^3$ were introduced in section 3, here we sum up their properties.

Properties of the relation R The basic property of R:

$$\text{if } Rxyz,\ x' \trianglelefteq x,\ y' \trianglelefteq y, \text{ and } z \trianglelefteq z', \text{ then } Rx'y'z'.$$

This means that the relation R is monotonic with respect to the involvement relation.

Moreover it is required that:

(r1) $Rxyz$ implies $Ryxz$

(r2) $R^2(xy)zw$ implies $R^2(xz)yw$, where R^2xyzw iff $(\exists s)(Rxys$ and $Rszw)$.

(r3) $Rxxx$

(r4) $Rxyz$ implies Rxz^*y^*

Properties of the relation C Compatibility between two states is inherited by the states involved in them ('less informative states'):

$$\text{If } xCy,\ x_1 \trianglelefteq x, \text{ and } y_1 \trianglelefteq y, \text{ then } x_1Cy_1.$$

Moreover, we require the following properties:

(c1) symmetricity xCy implies yCx

(c2) directedness $(\forall x)(\exists y)(xCy)$

(c3) convergence $(\forall x)(\exists y(xCy)$ implies $(\exists x^*)(xCx^*$ and $\forall z(xCz$ implies $z \trianglelefteq x^*)))$

(c4) $x \trianglelefteq y$ implies $y^* \trianglelefteq x^*$

(c5) $x^{**} \trianglelefteq x$

Model R-model **M** is a R-frame **F** with a valuation function $v : \mathcal{P} \longrightarrow 2^S$. The truth of a formula at a situation is defined in the following way:

- $s \Vdash p$ iff $s \in v(p)$
- $s \Vdash \neg\varphi$ iff $s^* \nVdash \varphi$
- $s \Vdash \psi \wedge \varphi$ iff $s \Vdash \psi$ and $s \Vdash \varphi$
- $s \Vdash \psi \vee \varphi$ iff $s \Vdash \psi$ or $s \Vdash \varphi$
- $s \Vdash (\varphi \to \psi)$ iff $(\forall y, z)(Rsyz$ implies $(y \Vdash \varphi$ implies $z \Vdash \psi))$

As we already said, the truth of a formula in a model, resp. in a frame, is defined as truth in all logical situations of this model/frame. As usual, R-tautologies are formulas true in all relevant frames. Whenever φ is a R-tautology, we write $\models \varphi$ and say that φ is a valid formula.

The condition (r1) validates the implicative version of *Modus Ponens* (axiom schema 3). It does not validate the conjunctive version $(A \wedge (A \to B)) \to B$, which requires (r3). (r2) corresponds to the *exchange rule* $(A \to (B \to C)) \to (B \to (A \to C))$, which is derivable from the axioms given above. (r4) validates *contraposition* (axiom schema 12). If we work without the Routley star, this can be rewritten as: $Rxyz$ implies $(\forall z'Cz)(\exists y'Cy)(Rxy'z')$.

Directedness and convergence conditions are necessary for the definition of the Routley star. From (c1) we obtain the validity of $(A \to \neg\neg A)$ and from the last condition (c5) we get the axiom schema 11.

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