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Epistemic logics for sceptical agents

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Abstract

In this article, we introduce an epistemic modal operator modelling knowledge over distributive non-associative full Lambek calculus with a negation. Our approach is based on the relational semantics for substructural logics: we interpret the elements of a relational frame as information states consisting of collections of data. The principal epistemic relation between the states is the one of being a reliable source of information, on the basis of which we explicate the notion of knowledge as information confirmed by a reliable source. From this point of view it is natural to define the epistemic operator formally as the backward-looking diamond modality. The framework is a generalization and extension of the system of relevant epistemic logic proposed by Majer and Peliš (2009, college Publications, 123–135) and developed by Bílková *et al.* (2010, college Publications, 22–38). The system is modular in the sense that the axiomatization of the epistemic operator is sound and complete with respect to a wide class of background logics, which makes the system potentially applicable to a wide class of epistemic contexts. Our system admits a weak form of logical omniscience (the monotonicity rule), but avoids stronger ones (a necessitation rule and a K-axiom) as well as some closure properties discussed in normal epistemic logics (like positive and negative introspection). For these properties we provide characteristic frame conditions, so that they can be present in the system if they are considered to be appropriate for some specific epistemic context. We also prove decidability of the weakest epistemic logic we consider, using a filtration method. Finally, we outline further extensions of our framework to a multiagent system.

Keywords: Epistemic logic, substructural logic, frame semantics.

1 Introduction

Mainstream approaches to epistemic logics are traditionally based on the notion of possible worlds. They interpret knowledge as a normal necessity-like operator over epistemic alternatives—knowledge is truth in all epistemic alternatives. The advantage of this approach is that it is simple and straightforward; moreover, it can use a well-developed apparatus of modal logics. However, as has been pointed many times in the literature, epistemic agents represented by normal modal logics are too perfect. They are logically omniscient because of the normality of the system, and, above that, in the most popular $S5$-like systems they are both positively and negatively introspective. Our aim is to provide a framework for less ideal agents applicable in a wide range of epistemic situations. We propose a general framework that makes only minimal assumptions about information it represents. This leads us to the choice of a weak background logic, extensions of which have a chance to cover a broad range of potential interpretations.

Instead of possible worlds we base our approach on the more general notion of *information state*. Information states are collections of pieces of information expressed by propositions. Unlike possible worlds they are typically incomplete (some information is missing) and can be inconsistent (some pieces of information contradict each other). We can see information state as a collection of data

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in some database—no database contain information about everything and databases might typically contain corrupted data that yield inconsistent information.

Obviously not every available piece of information is knowledge. Knowledge should be at least true and consistent and data in a database might violate both of these requirements. What should be the criteria for accepting some information as knowledge? One of the essential requirements for knowledge says that it should be *justified*. There is no explicit representation of justification in the mainstream approach, it is somehow implicitly embedded in the notion of epistemic alternatives or accessibility. There does not seem to be a straightforward analogue of a notion of epistemic alternatives in the framework based on information states, nor it seems one is needed. Instead we may take the notion of justification seriously and base our approach on justifications represented by information states itself, connected by an explicit relation to the current state. Let us stress that there is nothing essential in the background of the connection between knowledge and necessity; it is purely conventional. Quantifying universally makes sense over epistemic alternatives, but as there is no analogy to the relation of epistemic alternative in our framework, there is no reason to stick to the traditional reading of knowledge and associate knowledge with any kind of necessity operator.

To motivate our approach we will narrow for the moment the scope of epistemic situations we have in mind and concentrate on the area of scientific knowledge and its justification.

A prototypical situation will be the one of a scientist working in a laboratory. We can see scientist's own experimental activities in the laboratory and conclusions he/she eventually draws from them as his/her primary source of data. Obviously he/she has to confront his/her results with the results of the other researchers in the area. She also has to use secondary sources of data—relevant literature, scientific databases, conferences, etc. Scientific data as a whole are not in an ideal correspondence. Empirical methods suffer from the distortion and errors. Even if empirical data are correct, researchers might draw wrong hypotheses based on them. It is evident that sources of data can be highly problematic. They might contain parts conflicting each other or parts incompatible with data from another sources. Collections of scientific data (as well as most of the other kinds of data) are typically *incomplete* and might be *inconsistent*.

According to the standard scientific practice a scientist is supposed to be careful and to check the reliability of data before he/she accepts them as a scientific knowledge. The reliability depends, for example, on controlled conditions of an experiment (in case of prime data) or on the scientific reputation of the authors of secondary data. After the process of collection and reliability checking the scientist decides which data he/she accepts as scientific knowledge—information which can be responsibly spread in the scientific community. In the simplest case, he/she tries to find out if the same experiment with the same results was already performed by someone else, who is a trustable member of the community and hence is a *reliable source* of information. Our scientist accepts his/her results as a scientific knowledge only if it is *confirmed* by a reliable source. The notion of a source will be a central notion of our approach. Obviously we do not claim to cover the whole range of explications of the term 'justification'. There are various kinds of justification discussed in the literature and our approach is one possible solution.[1] One of the advantages is that this solution has a simple and intuitive characterization (our modality is factive, consistent, monotonic with respect to implication and it distributes over disjunction—see Definition 3.3) which are properties plausible in a wide range of epistemic contexts.

[1]The idea of an explicit representation of the notion of justification in a logical framework is not new. Sergei Artemov proposed a system of justification logic (cf. [1]), in which he can syntactically represent justification of a particular piece of knowledge. Our approach to justification is different—justification is explicated via an relation in the semantics of the background system.

Our careful scientist is a prototype of what we shall call a *sceptical agent*. In the theory of knowledge, *scepticism* means that there is no absolute certainty about existence of 'real world'. We use the term in narrower sense: as being careful when accepting available information.

Reasoning capabilities of epistemic agents we would like to represent are determined by the properties of the epistemic operator we shall propose together with the properties of the background logic. Which properties are adequate for the agent? There are several ways to reply to this question. Either we can choose one particular epistemic framework and argue for its universality or at least for adequacy in epistemic contexts we are focused on. We can also fix a background logic and discuss (in)adequacy of particular properties of the epistemic operator (recall, e.g. discussions about introspection in the epistemic versions of normal modal logics). In our previous article [4], we argued against the rule of weakening and used relevant logic as our background system. There are, however, reasons for questioning some other structural rules as well: a detailed discussion can be found in [14].

In this article, we propose a more general and more flexible pluralistic solution. Instead of arguing for particular properties of a particular framework and discussing its adequacy for particular situations, we present a modular system of epistemic frameworks that allows for adding/deleting various properties of both the epistemic operator and the background logic. A choice of a particular configuration of these properties can then be tailored for a particular epistemic situation. We start with a weak background logic and show that the axiomatization of the epistemic operator is sound and complete with respect to extensions of the background system with various combinations of structural rules. Structural rules change in various ways the properties of the background logic of information states and allow to adapt it for various kinds of information.

2 Substructural epistemic frames

The basic background logic we start with is the distributive non-associative full Lambek calculus DFNL [5], extended with a negation. We consider it so far the weakest logical framework we extend with our epistemic modality. As we already mentioned, various epistemic contexts might justify presence or absence of various structural rules. We need not assume any structural rules to hold in our basic framework at the beginning but we allow adding them to the basic framework if required by a particular application. The definition of the epistemic modality remains unchanged. Our goal is to provide a toolkit rather than an universal tool and allow a fine tuning for particular epistemic situations.

However, for the sake of a simpler presentation and better readability, we restrict ourselves from the beginning to the commutative full Lambek calculus DFNLe, thus assuming the rule of exchange, and as the result dealing with only one implication instead of two. Another restriction we adopt is having only a single negation, thus assuming that the compatibility relation is symmetric, and consequently that $\varphi \Vdash \neg\neg\varphi$ is valid. These restrictions are inessential from the technical point of view and all proofs we present in this article work in the weaker logic with two implications and two negations as well.

We use the following propositional language $\mathcal{L}$, built over a fixed set $\mathtt{Prop}$ of propositional letters, obtained from the language of the full Lambek calculus adding a negation and a modality K.

$$\varphi ::= p \mid \varphi \otimes \varphi \mid \varphi \to \varphi \mid \top \mid \bot \mid \mathtt{t} \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid K\varphi$$

We use the standard notion of substitution as a map $\sigma : \mathcal{L} \to \mathcal{L}$, generated by any map $\sigma_0 : \mathtt{Prop} \to \mathcal{L}$ inductively on the structure of a formula.

We follow Restall's book *An Introduction to Substructural Logics* [15] for most of this article. We assume the reader is familiar with Chapter 11—Frames I: Logics with Distribution. The frame semantics for distributive substructural logics is defined as follows:

DEFINITION 2.1
A *basic substructural frame* is a tuple $F_0 = (W, L, \leq, C, R)$, where W is a non-empty set of *states* and

- $\leq$ is a partial order on W.
- R is a ternary relation on W satisfying the following monotonicity condition:

$$Rxyz \text{ and } x' \leq x, y' \leq y, z' \geq z \text{ implies } Rx'y'z', \tag{1}$$

  and the following commutativity condition:

$$Rxyz \text{ implies } Ryxz. \tag{2}$$

- L, the set of *logical states*, is a non-empty upwards closed subset of $(W, \leq)$, satisfying

$$x \leq y \text{ iff there is } z \in L \text{ such that } Rzxy. \tag{3}$$

- C is a binary *compatibility* relation on W satisfying the following monotonicity condition:

$$xCy \text{ and } x' \leq x \text{ and } y' \leq y \text{ implies } x'Cy', \tag{4}$$

  and symmetricity:

$$xCy \text{ implies } yCx. \tag{5}$$

The partial order is the information order, the ternary relation R determines the meaning of the connectives fusion $\otimes$ and implication $\rightarrow$, and the compatibility relation C is used to interpret the negation $\neg$. The set L of logical states determines validity in a frame, and it provides an interpretation of the truth constant $\mathtt{t}$.

Conditions (1) and (4) are monotonicity conditions, they require that the corresponding relation is preserved along the ordering $\leq$ and they correspond to the monotonicity (or distribution) type of the respective connectives they interpret. Condition (5) expresses the symmetry of the compatibility relation and it entails that we can deal with a single negation. Likewise, condition (2) expresses a commutativity of R and it entails that the fusion connective is commutative (i.e. we assume that the rule of exchange is valid).

Now we are ready to introduce the principal part of our framework—a binary relation S of 'being a source', on which we base the interpretation of our epistemic modality. We read xSy as 'the state x is a reliable source of information for the state y'.

The substructural framework we just introduced nicely corresponds to our example of working scientist we mentioned, in the previous section. While the scientist can access various collections of data, some of them are of a special importance for his/her research in the sense that he/she wants to be in agreement with them—-e.g. results of top research institutions in the field—those are the collections of data he/she wants to be *compatible* with. In particular he/she does not want to claim, that some statement is false if a compatible state 'claims' that it is true. It is clear from the example that compatibility relation might be in general non-symmetric and states compatible with the current state are in general different than the states which the current one is compatible with.

The amount of scientific data increases along the time. Even empirical data that has gotten obsolete and hypotheses that have been rejected, are very often stored anyway (e.g. to avoid similar kind of errors in future). We assume that the collections of data in our model grow, the later states contain more information than the earlier ones and no information is lost in this process.

To sum up our scientific agent has to deal with large amounts of data some of them being contradictory or corrupted in some other way. She has to be very careful when deciding which kind of data he/she can use. Our agent is sceptical—he/she accepts a particular piece of data as knowledge only if it is *confirmed* by a *source*. In our interpretation it means there is a reliable source (a collection of data) in which this piece of data is true. The relation of 'being a reliable source' plays a crucial role in our epistemic framework. There are two basic conditions we impose on the relation of a reliable source—it shall be a collection of data the agent trusts in the sense he/she wants to be compatible with it, and it must precede his/her current information state in the sense that all the data it contains should be available to his/her in the current state. It means we admit situations when an information state is a source for itself. This might be reasonable in some cases e.g. when the current state corresponds to results of a big scientific team. In some other cases, it would be more adequate to require that the source is different from the current state (we can see it as some sort of 'independence' of confirmation). However, this requirement introduces a technical problem—the class of the corresponding frames is not definable. We understand the conditions of compatibility and (non-strict) precedence as necessary, but not sufficient, hence not everything which could be a source is actually a source. If we make these conditions sufficient as well, we obtain again a non-axiomatizable theory. We present both the undefinability results in Section 2.3.

Definition 2.2

A *basic epistemic frame* is a couple $F=(F_0,S)$ such that F_0 is a basic substructural frame and S is a binary *source* relation on W satisfying the conditions

$$sSx \text{ implies } s\leq x \tag{6}$$

$$sSx \text{ implies } xCs \tag{7}$$

$$x'\leq x \text{ and } y\leq y' \text{ and } xSy \text{ implies } x'Sy' \tag{8}$$

The first condition (6) says that a source for a current state must precede the current state in the underlying ordering of the frame, meaning that the information of the source must be available at the current state, and the second condition (7) says that the current state must be compatible with a source, meaning that a reliable source cannot contradict the current state. The third condition (8) is again a monotonicity condition, and it guarantees that the backward-looking diamond modality it interprets is a monotone modality, and that the meaning of modal formulas is persistent.

We define an epistemic modality K of knowledge, reading $K\varphi$—'an agent knows φ'— as 'φ is confirmed by a reliable source'. It differs from the normal epistemic modalities of knowledge in several aspects. Main difference is that K is a diamond-like modality, existentially quantifying over the available sources of information, rather then a box-like modality universally quantifying over the epistemic alternatives. Another difference is that K is 'backward-looking' along a relation of 'being a source for' rather then forward-looking along a relation of 'being accessible from'. For a piece of information to become known in a state, we require the information to be confirmed by a source connected to the state.

Example 2.3

We illustrate our notion of knowledge on a simple model (Figure 1).

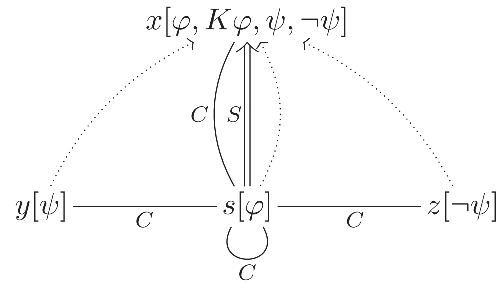

FIGURE 1. The dotted arrows depict the partial order. It holds $y \Vdash \psi, z \Vdash \neg\psi$ and $x \Vdash \varphi, K\varphi, \psi, \neg\psi$.

The actual state of the scientist is x. The state s plays the role of a source-state—data available at s precede those at the (actual) state x, moreover they are compatible with the data at x. The scientist accepts φ as knowledge in the state x, because there is a source-state confirming φ. States y and z are not sources, they precede the actual state x, but they are not compatible with it, therefore both ψ and $\neg\psi$ are data available at x, but neither of them is accepted as knowledge. Observe, that C is not transitive in our example—y is compatible with s, which is compatible with z, but y is not compatible with z.

Epistemic models are defined in a standard way as frames equipped with a persistent valuation of propositional formulas.

DEFINITION 2.4
A *basic epistemic model* is $M = (F, V)$, where F is a basic epistemic frame and $V : \mathtt{Prop} \mapsto \mathcal{U}(W)$ is a valuation where $\mathcal{U}(W)$ is the set of upward closed subsets of $(W, \leq)$. The valuation is persistent—if $x \in V(p)$ and $x \leq y$ then $y \in V(p)$. It generates the following satisfaction relation between states and formulas:

- $x \Vdash p$ iff $x \in V(p)$
- $x \Vdash \mathtt{t}$ iff $x \in L$
- $x \Vdash \top$
- $x \nVdash \bot$
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$
- $x \Vdash \varphi \vee \psi$ iff $x \Vdash \varphi$ or $x \Vdash \psi$
- $x \Vdash \neg\varphi$ iff for all y, xCy implies $y \nVdash \varphi$
- $x \Vdash \varphi \to \psi$ iff for all y, z, $Rxyz$ and $y \Vdash \varphi$ implies $z \Vdash \psi$
- $x \Vdash \varphi \otimes \psi$ iff there exist y, z such that $Ryzx$ and $y \Vdash \varphi$ and $z \Vdash \psi$
- $x \Vdash K\varphi$ iff there exists s such that sSx and $s \Vdash \varphi$

We use the following notions of validity and consequence:

- Validity in a model: $M \Vdash \varphi$ iff $(\forall x \in L)\, M, x \Vdash \varphi$.
  Thus, validity in a model is defined as truth in all *logical* states.
- Validity in a frame: $F \Vdash \varphi$ iff $\forall V (F, V) \Vdash \varphi$.
- (Local) consequence in a model: $\Gamma \vDash_M \psi$ iff $\forall x (M, x \Vdash \Gamma \Rightarrow M, x \Vdash \psi)$, where $M, x \Vdash \Gamma$ stays for $(\forall \gamma \in \Gamma) M, x \Vdash \gamma$. Similarly we can define local consequence in a frame and a class of frames: $\Gamma \vDash_F \psi$ iff for all models M over the frame F it is the case that $\Gamma \vDash_M \psi$, and $\Gamma \vDash_{\mathcal{F}} \psi$ iff for all $F \in \mathcal{F}$ it is the case that $\Gamma \vDash_F \psi$.

- A rule $\varphi_1,\ldots,\varphi_n/\psi$ is valid in a frame F iff for all substitutions σ, $\forall i\ F\Vdash\sigma(\varphi_i)\Rightarrow F\Vdash\sigma(\psi)$.
- Writing $\Gamma\vDash\psi$ or $\varphi\vDash\psi$ without the index we mean consequence in the class of all frames.

Observe that, thanks to the condition (3), $\varphi\vDash_M\psi$ holds if and only if $M\Vdash\varphi\to\psi$, thus the simple consequence coincides with validity of the corresponding implication.

LEMMA 2.5 (PERSISTENCY FOR FORMULAS)
Let F be a basic epistemic frame. Then for any state $x, x\Vdash\varphi$ and $x\le y$ implies $y\Vdash\varphi$.

PROOF. We concentrate on the case of $K\varphi$ only. Suppose that $x\Vdash K\varphi$ and $x\le y$. We show that $y\Vdash K\varphi$. Since $x\Vdash K\varphi$, there is some sSx satisfying φ. From condition (8) applied on $s\le s$, $x\le y$ and sSx we obtain sSy, then s is a source for y too, thus, $y\Vdash K\varphi$ as desired. ■

Relational semantics of substructural logics allows a special reading of implication. Let us return to the example of a scientist working in a laboratory. She is performing experiments and observations and gets empirical data from which he/she would like to derive some hypotheses and theories about the observed phenomena. The scientist is in particular interested in correlations between results of observation from which he/she can hypothesize about causal relations between observed phenomena. One way to express a hypothesis about a correlation is an *association rule*,[2] which can be presented in the form of implication. Assume the scientist is making an experiment concerning the relation of electricity and magnetism. She starts with some initial data φ (a wire carrying an electric current) and when the experiment is finished, he/she observes the result ψ of the experiment (there is an indication of a magnetic field around the wire). If these observations appear regularly in different instances of the experiment, he/she formulates a hypothesis in the form of an association rule: If (I observe initial data—an electric current in a conductor) φ, then (they are followed by resulting data—a magnetic field is generated) ψ.

2.1 Basic properties of sources and knowledge

Let us continue with observing some general properties of the source relation and the epistemic modality K. Namely, all sources for a state are mutually compatible as well as self-compatible, and therefore they are mutually consistent.

LEMMA 2.6 (PROPERTIES OF SOURCES)
Let F be a basic epistemic frame. Then for any s, x:

(1) sSx implies sCs (self-compatibility of sources)
(2) sSx implies, that there is no φ such that $s\Vdash\varphi$ and $s\Vdash\neg\varphi$ (consistency of sources)
(3) sSx and $s'Sx$ implies sCs' (mutual compatibility of sources)
(4) $x\Vdash K\varphi$ then for any $sSx, s\nVdash\neg\varphi$ (consistency of knowledge with respect to sources)

PROOF. (i) sSx implies $s\le x$ and xCs. From the condition (4) it follows that sCs, which immediately implies (ii). (iii) From sSx we have $s\le x$ which together with xCs' yields sCs'. (iv) follows immediately from (iii). ■

Consistency and mutual compatibility of sources guarantee that our concept of confirmation by a source yields consistent knowledge, as we shall see in the Lemma 2.7.

[2]The term *association rule* comes from data mining.

LEMMA 2.7 (PROPERTIES OF K)
Let F be a basic epistemic frame. Then the following formulas are valid in F:

(1) $K\varphi \to \varphi$ (truth)
(2) $\neg\varphi \wedge K\varphi \to \bot$ (consistency)
(3) $(\varphi \to \psi)/(K\varphi \to K\psi)$ (monotonicity)
(4) $K(\varphi \vee \psi) \to (K\varphi \vee K\psi)$ ($\vee$-distributivity)

PROOF. (i) Truth scheme: $K\varphi \to \varphi$ holds, since, whenever sSx and $s \Vdash \varphi$ then by the condition (6) $s \leq x$ and thus $x \Vdash \varphi$.

(ii) Consistency scheme: note that $x \Vdash \psi \to \bot$ iff $x \nVdash \psi$. Now if $x \Vdash \neg\varphi \wedge K\varphi$, for some x then there is an s such that $sSx, s \Vdash \varphi$, but from the definition of S we have xCs; since $x \Vdash \neg\varphi$, we get $s \nVdash \varphi$, a contradiction. So there is no such x where $\neg\varphi \wedge K\varphi$ holds.

(iii) Monotonicity rule: consider $x \in L$ and y, z, such that $Rxyz$ and $y \Vdash K\varphi$. We show that $z \Vdash K\psi$. $y \Vdash K\varphi$ implies there is s such that (sSy and $s \Vdash \varphi$). $s \leq s$ implies there is $t \in L$ such that $Rtss$ (by the condition for L). Since in such $t \Vdash \varphi \to \psi$ and $s \Vdash \varphi$, we conclude $s \Vdash \psi$. Thus $y \Vdash K\psi$. But $y \leq z$ (since $x \in L$ and $Rxyz$), and finally $z \Vdash K\psi$.

(iv) $\vee$-distributivity scheme: suppose $x \in L$, $Rxyz$ and $y \Vdash K(\varphi \vee \psi)$ we show that $z \Vdash K\varphi \vee K\psi$. There is sSy satisfying $\varphi \vee \psi$, suppose, e.g., $s \Vdash \varphi$. From $Rxyz$ and $x \in L$ we know $y \leq z$. By the condition (6) there is s' such that $s \leq s'$ and $s'Sz$. Now $s' \Vdash \varphi$ and $z \Vdash K\varphi$. Then $z \Vdash K\varphi \vee K\psi$. ∎

The truth scheme is usually taken as a constitutive property of knowledge, in contrast to belief. In the classical modal frameworks, the truth scheme is derivable from the scheme of consistency and vice versa, but in substructural logics it is not always the case. We would need to use additional axioms concerning negation to relate the two schemes, namely, in presence of $\varphi \wedge \neg\varphi \vdash \bot$ truth derives consistency, and in the presence of $\top \vdash \varphi \vee \neg\varphi$ consistency derives truth (see Example 3.4 in the following section).

Admitting the $\vee$-distributivity means that, in the context of knowledge, we read disjunction constructively: in any information state, an agent knows a disjunction only if he/she knows one of the disjuncts. This property is problematic as we can imagine epistemic situations in which it fails (certainly, one may know that there is a finite number of possible explanation of observed phenomena without being able to say which one is confirmed). The property is a consequence of interpreting a confirmation of a piece of information as a whole information state, and of our choice of the background logic being distributive and, therefore, the information states being prime theories closed under disjunction. Since the knowledge modality is a backward-looking diamond-like operator and as such it is a left adjoint, it automatically distributes with disjunctions.

As we already pointed out, the monotonicity rule is also problematic, as it is considered to be one form of logical omniscience. However, as we want to have a knowledge operator with at least some closure properties with respect to the background logic, this is the minimal requirement we can have. Let us note that our operator does not in general distribute over conjunction and hence our monotonicity is weaker than that of standard operators: from the fact that $\Gamma \vDash \varphi$, for a finite set of formulas Γ, we cannot in general derive $K\Gamma \vDash K\varphi$, where $K\Gamma = \{K\gamma \mid \gamma \in \Gamma\}$, but only $K(\bigwedge \Gamma) \vDash K\varphi$. There are other, stronger forms of logical omniscience, which our system avoids—namely a scheme corresponding to the K axiom (distributivity to the implication) and the necessitation rule of standard modal logics. Instead of simply showing that the logic of basic epistemic frames we have just defined avoids them, we give frame conditions on the source relation S characterizing the frames validating these principles, and a few other epistemic axioms and rules.

2.2 *Definability of some classes of epistemic frames*

We say that a class of frames $\mathcal{F}$ is *characterized*, or *defined*, by a formula (or by a set of formulas) if and only if $\mathcal{F}$ is the class of all frames validating the formula (or the set of formulas). We say that a class of frames $\mathcal{F}$ is characterized, or defined, by a rule if and only if $\mathcal{F}$ is the class of all frames validating the rule (i.e. the class of frames in which the validity is closed under all the instances of the rule).

LEMMA 2.8 (CHARACTERIZATION OF SOME MODAL PROPERTIES OF K)

name	axiom or rule	frame condition
necessitation	$\varphi/K\varphi$	$(\forall x\in L)\ (\exists s\in L)(sSx)$
$\to$-distribution	$K(\varphi\to\psi)\vdash(K\varphi\to K\psi)$	$(Rxyz\wedge sSx\wedge tSy)\to\exists w(wSz\wedge Rstw)$
$\otimes$-distribution 1	$K\varphi\otimes K\psi\vdash K(\varphi\otimes\psi)$	$(Rxyz\wedge sSx\wedge tSy)\to\exists w(wSz\wedge Rstw)$
$\otimes$-distribution 2	$K(\varphi\otimes\psi)\vdash K\varphi\otimes K\psi$	$(Ruvs\wedge sSz)\to\exists x\exists y(uSx\wedge vSy\wedge Rxyz)$
$\wedge$-distribution	$K\varphi\wedge K\psi\vdash K(\varphi\wedge\psi)$	$sSx\wedge tSx\to\exists v(vSx\wedge s\leq v\wedge t\leq v)$
positive introspection	$K\varphi\vdash KK\varphi$	$sSx\to\exists y(sSySx)$
negative introspection	$\neg K\varphi\vdash K\neg K\varphi$	$\exists s(sSx\wedge\forall u\forall v(uSv\wedge sCv\to\exists y\exists z(u\leq y\wedge ySz\wedge xCz)))$

PROOF.
Necessitation. The condition states that each logical state has a logical source. Let us have a frame F satisfying the frame condition and fix a state $x\in L$ and fix a valuation. By the condition there is some $s\in L$ such that sSx. If φ is valid, it holds in every frame in every logical state, hence $x\Vdash\varphi$ but $s\Vdash\varphi$ too and s is a source for x, hence $x\Vdash K\varphi$ as required.
Consider a frame for which the frame condition does not hold. In particular, there is some $x\in L$ such that there is no $s\in L$ with sSx. Consider the truth constant $\mathtt{t}$. By definition it holds in all logical states, but nowhere else. So we have $\Vdash_F \mathtt{t}$. As x is logical, $x\Vdash\mathtt{t}$. But x has only non-logical sources (if any), none of them can confirm t, so $x\nVdash K\mathtt{t}$ hence $\nVdash_F K\mathtt{t}$ as required.

$\to$**-distribution**. Let us have a frame F satisfying the condition and fix a valuation. Let $x\Vdash K(\varphi\to\psi)$, therefore some sSx satisfies $s\Vdash\varphi\to\psi$. Now let $Rxyz$ where $y\Vdash K\varphi$, thus for some tSy we have $t\Vdash\varphi$. By the condition there is some w satisfying $Rstw$, thus in particular $w\Vdash\psi$. By the condition also wSz, thus $z\Vdash K\psi$ as required.
Consider a frame for which the frame condition does not hold, i.e. it contains a situation $Rxyz$ and sSx and tSy which violates the condition. We define the following valuation: $V(p)=\{w\mid w\geq t\}$ and $V(q)=\{w\mid Rstw\}$. Both are upward closed sets and we obtain $t\Vdash p$ and $s\Vdash p\to q$, and therefore $x\Vdash K(p\to q)$ and $y\Vdash Kp$. Since all the possible sources w for z violate the condition $Rstw$, neither of them satisfies q. Thus $z\nVdash Kq$ as desired.

$\otimes$**-distribution.** The first direction of the $\otimes$-distribution defines the same class of frames as $\to$-distribution, and the proof of this fact is similar to the one given above, using the residuation law. We concentrate to the other direction of the $\otimes$-distribution.
Consider a frame satisfying the condition and fix a valuation. Let $z\Vdash K(\varphi\otimes\psi)$, therefore for some sSz we have $s\Vdash\varphi\otimes\psi$ and there are $Ruvs$ with $u\Vdash\varphi$ and $v\Vdash\psi$. By the condition there are x and y with $Rxyz$ and uSx and vSy, thus $x\Vdash K\varphi$ and $y\Vdash K\psi$. Therefore $z\Vdash K\varphi\otimes K\psi$.
Consider a frame violating the frame condition, i.e. there is a situation $RuvsSz$ which violates the condition. We define the following valuation: $V(p)=\{w\mid u\leq w\}$ and $V(q)=\{w\mid v\leq w\}$, thus satisfying $z\Vdash K(p\otimes q)$. Now by u and v violating the condition, no states x,y with uSx and vSy satisfy $Rxyz$. Observe that, therefore, by monotonicity of S, no states x,y with $x\Vdash Kp$ and $y\Vdash Kq$ satisfy $Rxyz$, proving that $z\nVdash Kp\otimes Kq$.

$\wedge$-distribution. The frame condition states that sources have upper bounds. Let us have a frame that satisfies this condition, fix a valuation and consider some $x \Vdash K\varphi \wedge K\psi$. Then there are sSx and tSx satisfying $s \Vdash \varphi$ and $t \Vdash \psi$. By the condition, there is an upper bound v satisfying vSx, but then $v \Vdash \varphi \wedge \psi$ and we obtain $x \Vdash K(\varphi \wedge \psi)$.
Consider a frame violating the condition, i.e. there is some x with sSx and tSx and s,t do not have an upper bound which is a source for x. Define a valuation as follows: $V(p) = \{w \mid s \leq w\}$ and $V(q) = \{w \mid t \leq w\}$. Now any possible source for x satisfying $p \wedge q$ would be an upper bound of s and t, thus no such source exists. Therefore $x \nVdash Kp \wedge Kq$.

Positive introspection. The condition states that the S relation is dense. Assume a frame satisfying the condition, fix a valuation and consider some $x \Vdash K\varphi$ via some sSx with $s \Vdash \varphi$. By the condition there is some y with $sSySx$. Then $y \Vdash K\varphi$ and $x \Vdash KK\varphi$.
Consider a frame violating the condition, i.e. it contains a state sSx without $sSySx$ for any y. Define a valuation as follows: $V(p) = \{w \mid s \leq w\}$. Thus $x \Vdash Kp$, while by monotonicity of S any $w \geq s$ and any y violate $wSySx$, thus there is no source $w \Vdash p$ with $wSySx$, and therefore KKp is refuted in x.

Negative introspection. Assume a frame satisfying the condition, fix a valuation and consider an $x \Vdash \neg K\varphi$. We want to see that $x \Vdash K\neg K\varphi$. By the frame condition there is an sSx. Consider any v with sCv, we show it does not satisfy $K\varphi$: to this end consider any uSv and see that u does not satisfy φ, which in turn means that $s \Vdash \neg K\varphi$. If u did satisfy φ, by the condition there must be some $y \geq u$ (thus satisfying φ as well) with ySz and xCz for some z. But then $z \Vdash K\varphi$ and this contradicts $x \Vdash \neg K\varphi$. Thus $u \nVdash \varphi$ and $s \Vdash \neg K\varphi$ as desired.
Consider a frame violating the condition, meaning there is an x with no source s satisfying what the condition prescribes. Define a valuation $V(p)$ negatively by its complement $W - V(p) = \{w \mid \exists z(wSz \wedge xCz)\}$, which is the complement of a downward closed set, thus an upperset. Therefore, $x \Vdash \neg Kp$ because no state compatible with x has a p-source. We show that $x \nVdash K\neg Kp$. Consider any sSx and see that it refutes $\neg K\varphi$. Since the frame condition is violated, there must be some v and u with sCv and uSv such that u is not below any y with ySz and xCz, thus in particular, u is not below any state refuting p. But that means that $u \Vdash p$ and consequently $v \Vdash Kp$ and $s \nVdash \neg Kp$. ■

In the following section, after proving completeness, we show that all the axioms mentioned above are canonical—they hold in their respective canonical frames.

2.3 *Examples of undefinability*

In the earlier paper [4], we discussed some particular classes of epistemic frames—the weak classic frames (all compatible states below the current state are sources for the state) and the classic frames (sources are moreover strictly below the current state). Using frame morphisms we can show that they are not characterizable.We show that a bit weaker condition $sSx \rightarrow s < x$ is undefinable, but the undefinability of the two mentioned classes clearly follows. The undefinability of a class means that every axiomatization complete with respect to the class will also have models outside the class.

We adopt the notion of frame morphism given in [3], taking moreover in account that the modality K is backward-looking.

Definition 2.9 (Frame morphisms)
A *frame morphism* between frames $F_1 = (W_1, L_1, R_1, C_1, S_1)$ and $F_2 = (W_2, L_2, R_2, C_2, S_2)$ is a monotone map $f : W_1 \rightarrow W_2$ satisfying the following conditions:

- $f(x) \in L_2$ iff $x \in L_1$

- $R_2uvf(z)$ iff $\exists x, y \in W_1(u \leq f(x) \wedge v \leq f(y) \wedge R_1xyz)$
- $R_2f(x)vw$ iff $\exists y, z \in W_1(v \leq f(y) \wedge f(z) \leq w \wedge R_1xyz)$
- $uC_2f(x)$ iff $\exists y \in W_1(u \leq f(y) \wedge yC_1x)$
- $uS_2f(x)$ iff $\exists y \in W_1(u \leq f(y) \wedge yS_1x)$

In this more compact definition of morphisms than usual, the right–left directions of the items above imply that frame morphisms preserve the relations (the usual forth conditions), while the left–right directions are the usual back-conditions. Also, thanks to the fact that C is symmetric, we have covered the case when $f(x)C_2u$ as well, and the same applies to R being commutative.

LEMMA 2.10 (PRESERVATION OF FRAME VALIDITY)
Let $f: W_1 \to W_2$ be a surjective frame morphism from F_1 to F_2. Then for each formula φ in $\mathcal{L}$

$$F_1 \Vdash \varphi \Rightarrow F_2 \Vdash \varphi.$$

PROOF. Let V_2 be a valuation on F_2 refuting φ. Define a valuation on F_1 by $V_1(p) = \{x \mid f(x) \in V_2(p)\}$. Now it holds for every formula ψ that

$$F_1, V_1, x \Vdash \psi \text{ iff } F_2, V_2, f(x) \Vdash \psi.$$

This can be proved by a routine induction on the formula ψ, we spell out only some of the cases, namely for negation and the K modality. The remaining cases are similar.

Assume $F_1, V_1, x \Vdash \neg\alpha$. We show that $F_2, V_2, f(x) \Vdash \neg\alpha$. Consider any v such that $f(x)C_2v$. Then, by f being a frame morphism, there exists $y \in W_1$ such that $v \leq f(y)$ and xC_1y. By the assumption $F_1, V_1, y \nVdash \alpha$. By the induction hypothesis $F_2, V_2, f(y) \nVdash \alpha$, thus $F_2, V_2, v \nVdash \alpha$.

Assume $F_2, V_2, f(x) \Vdash \neg\alpha$. We show that $F_1, V_1, x \Vdash \neg\alpha$. Consider any y such that xC_1y. Then, by the condition, $f(x)C_2f(y)$ and by the assumption $F_2, V_2, f(y) \nVdash \alpha$. By the induction hypothesis $F_1, V_1, y \nVdash \alpha$.

Assume $F_1, V_1, x \Vdash K\alpha$. We show that $F_2, V_2, f(x) \Vdash K\alpha$. By assumption, there is some sS_1x with $F_1, V_1, s \Vdash \alpha$. By the condition $f(s)S_2f(x)$ and by the induction hypothesis $F_2, V_2, f(s) \Vdash \alpha$.

Assume now that $F_2, V_2, f(x) \Vdash K\alpha$. Thus, there is a u with $uS_2f(x)$ and $F_2, V_2, u \Vdash \alpha$. By the condition there exists a y satisfying $u \leq f(y) \wedge yS_1x$. Thus, in particular $F_2, V_2, f(y) \Vdash \alpha$. By the induction hypothesis $F_1, V_1, y \Vdash \alpha$, thus $F_1, V_1, x \Vdash K\alpha$.

Since F_2 refutes φ, there is a $u \in W_2$ with $F_2, V_2, u \nVdash \varphi$. Because f is surjective, there is an x such that $f(x) = u$, and therefore $F_1, V_1, x \nVdash \varphi$. ■

REMARK 2.11
Restall in the book [15] shows similar undefinability results using a concept of directed bisimulations. In the paper [3] a general definability theorem is proved for logics of distributive lattices with operators (not necessarily residuated) of which distributive full Lambek logics with a negation and a monotone modality like K are a special case. The theorem states that a class of frames, closed under prime extensions, is definable if and only if it is closed under generated subframes (i.e. reflects frame morphisms which are order embeddings), it is closed under disjoint unions of frames and frame-morphic images (i.e. surjective frame morphisms), and it reflects prime extensions. From the theorem, the closure under frame-morphic images applies here.

LEMMA 2.12
The following classes of frames are not characterizable:

(i) The class of frames satisfying $sSx \to s < x$ (sources are strictly below).
(ii) The class of frames satisfying $xCs \wedge s \le x \to sSx$ (all compatible states below are sources).

PROOF. (i) The class of frames satisfying $sSx \to s < x$: let F_1 be defined over integers with the strict order as follows: $W_1 = Z$, $\le_1 = S_1 = <_Z$ and C_1 is total. Let F_2 be defined as $W_2 = \{u\}$ with uC_2u and uS_2u. The argument does not depend on L and R, they can in both the frames be considered total, thus certainly satisfying the conditions required in the definition of a frame. F_1 is in the class of frames we consider, while F_2 is not. Let $f: F_1 \to F_2$ be defined as $f(n) = u$ for each $n \in Z$. Then f is a frame morphism, and F_2 is a morphic image of F_1. To see this we check the last two items in Definition 2.9: the right–left implication are trivial because $uC_2f(n)$ holds for each $n \in Z$, and similarly $uS_2f(n)$ holds for each $n \in Z$. Consider $n \in Z$, then $uC_2f(n)$ and we have to check the left–right implication: since $u \le f(n-1)$ and $(n-1)C_1n$, we conclude that $n+1$ witnesses the right side of the equation. Similarly for S_2: because $uS_2f(n)$ holds for each n, we have to witness the right side of the equation. But again $u \le f(n-1)$ and $(n-1)S_1n$ holds for each n.

(ii) The class of frames satisfying $(xCs) \wedge (s \le x) \to sSx$: let F_1 be defined as $W_1 = \{x, y\}$ where x and y are incomparable, and xC_1y and yC_1x. Let F_2 be defined as $W_2 = \{u\}$ and uC_2u. Relations S_1 and S_2 are empty in both frames. Again, L and R are total in both frames. F_1 is in the class of frames we consider (since $C_1 \cap \le$ is empty), while F_2 is not. Let $f: F_1 \to F_2$ be defined as $f(x) = f(y) = u$. Then f is a frame morphism, and F_2 is a morphic image of F_1. To see this we need to check the one but last item of Definition 2.9. The right–left direction is again trivially satisfied. For $uC_2f(x)$ we use y which satisfies $u \le f(y)$ and yC_1x. For $uC_2f(y)$ we use x which satisfies $u \le f(x)$ and xC_1y.

■

3 Axiomatization

We give a Hilbert-style axiomatization of the epistemic logic introduced semantically in the previous section. The calculus is defined as the appropriate extension of the axiomatization of the background substructural logic—the non-associative Lambek calculus with the axioms corresponding to the distributivity law, the exchange axiom, and the contraposition rule. We call this basic axiomatization $\mathcal{L}_0$.

DEFINITION 3.1 (AXIOMATIZATION OF $\mathcal{L}_0$)
The calculus consists of the following axiom schemes and rules (double lines denote two-way rules):

$$\varphi \to \varphi \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi} \qquad \frac{\varphi \to \psi \qquad \psi \to \chi}{\varphi \to \chi}$$

$$\frac{\varphi \to (\psi \to \chi)}{\overline{\psi \otimes \varphi \to \chi}} \qquad \frac{\varphi \to (\psi \to \chi)}{\overline{\psi \to (\varphi \to \chi)}}$$

$$\varphi \wedge \psi \to \varphi \qquad \varphi \wedge \psi \to \psi \qquad \frac{\chi \to \varphi \qquad \chi \to \psi}{\chi \to \varphi \wedge \psi}$$

$$\varphi \to \varphi \vee \psi \qquad \psi \to \varphi \vee \psi \qquad \frac{\varphi \to \chi \qquad \psi \to \chi}{\varphi \vee \psi \to \chi}$$

$$\varphi \to \top \qquad \bot \to \varphi \qquad \frac{\mathtt{t} \to \psi}{\psi}$$

$$\varphi \wedge (\psi \vee \chi) \to (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$\frac{\psi \to \neg\varphi}{\varphi \to \neg\psi}$$

EXAMPLE 3.2
For example, $\varphi \to \neg\neg\varphi$ is derivable in the calculus:

$$\frac{\neg\varphi \to \neg\varphi}{\varphi \to \neg\neg\varphi}$$

the rule of contraposition is derivable in the calculus:

$$\cfrac{\cfrac{\varphi \to \psi \qquad \psi \to \neg\neg\psi}{\varphi \to \neg\neg\psi}}{\neg\psi \to \neg\varphi}$$

Monotonicity of fusion can be proved as follows:

$$\cfrac{\cfrac{\varphi \to \psi \qquad \cfrac{\psi \otimes \chi \to \psi \otimes \chi}{\psi \to (\chi \to \psi \otimes \chi)}}{\varphi \to (\chi \to \psi \otimes \chi)}}{\varphi \otimes \chi \to \psi \otimes \chi}$$

It is not hard to show other basic properties as e.g. the neutrality of $\mathtt{t}$ w.r.t. the fusion connective, or that $\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$ and vice versa. We leave the rest for the reader.

We obtain the axiomatization of the logic of Lambek epistemic frames adding the following two axioms and a rule for the modality K to the background calculus $\mathcal{L}_0$:

DEFINITION 3.3 (AXIOMATIZATION OF $\mathcal{L}_0^K$)
The axiomatization of the logic of Lambek epistemic frames consists of the axioms for the background logic $\mathcal{L}_0$, the axiom schemes

Truth $K\varphi \to \varphi$
Consistency $\neg\varphi \wedge K\varphi \to \bot$
$\vee$-distribution $K(\varphi \vee \psi) \to K\varphi \vee K\psi$

and the rule

$$\frac{\varphi \to \psi}{K\varphi \to K\psi}$$

We write $\vdash \varphi$ for φ being derivable in the system. We write $\varphi \vdash \psi$ if and only if $\vdash \varphi \to \psi$ is derivable, and we write $\Gamma \vdash \Delta$ if and only if there are finite subsets $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ such that $\bigwedge \Gamma' \vdash \bigvee \Delta'$.

EXAMPLE 3.4
We show that, in the presence of $\top \vdash \varphi \vee \neg\varphi$, the truth scheme becomes derivable from the consistency schema. We give only a sketch of a derivation, skipping the easy steps:

$$\dfrac{\dfrac{\dfrac{\dfrac{K\varphi \wedge \neg\varphi \to \bot}{(K\varphi \wedge \neg\varphi) \vee \varphi \to \varphi}}{(K\varphi \vee \varphi) \wedge (\neg\varphi \vee \varphi) \to \varphi} \qquad \dfrac{\top \vdash \varphi \vee \neg\varphi}{(K\varphi \vee \varphi) \to (K\varphi \vee \varphi) \wedge (\neg\varphi \vee \varphi)}}{(K\varphi \vee \varphi) \to \varphi}}{K\varphi \to \varphi}$$

In the presence of $\varphi \wedge \neg\varphi \to \bot$, the consistency scheme becomes derivable from the truth scheme. We give a sketch of a derivation, again skipping the easy steps:

$$\dfrac{\dfrac{K\varphi \to \varphi}{(K\varphi \wedge \neg\varphi) \to \varphi \wedge \neg\varphi} \qquad \varphi \wedge \neg\varphi \to \bot}{(K\varphi \wedge \neg\varphi) \to \bot}$$

REMARK 3.5
A Hilbert-style axiomatization of the underlying logic—the non-associative Lambek calculus and some of its extensions can be found e.g. in [7, Ch. II.] or in [8], as the axiomatization of the basic substructural logic they call SL. However, their notion of a consecution does not coincide with our definition of $\Gamma \vdash \varphi$—it is rather interpreted as $\gamma_1 \otimes \ldots \otimes \gamma_n \to \varphi$ which is more usual and convenient as it fits well with the residuation laws. Our motivation for using $\Gamma \vdash \varphi$ for $\bigwedge \Gamma \to \varphi$ is that it fits with the notion of a pair and that it corresponds to the local consequence under which the information states are closed.

Our goal was to give a definition of an epistemic modality which conservatively extends a wide class of background logics. The following definition is in fact a scheme allowing to construct epistemic frames with given additional structural properties. In the table below, we mention some important structural axioms, namely those which make sense to add to the commutative distributive full Lambek calculus (e.g. in the absence of exchange, some of the structural properties have two forms—left and right—we can ignore this distinction here). For the proof that the axiom defines the class of the frames given by the corresponding condition in the table we refer to [15].

DEFINITION 3.6 (AXIOMATIZATION OF SUBSTRUCUTRAL FRAMES)

sign	name	axiom	condition
a	associativity	$(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$	$R(xy)zw \to Rx(yz)w$
c	contraction	$(\varphi \to (\varphi \to \psi) \to (\varphi \to \psi)$	$Rxyz \to R(xy)yz$
c'	weak contraction	$(\varphi \to \psi) \wedge \varphi \to \psi$	$Rxxx$
m	mingle	$\varphi \to (\varphi \to \varphi)$	$Rxxy \to x \leq y$
w	weakening	$\varphi \to (\psi \to \varphi)$	$Rxyz \to x \leq z$
λ	l.e.m.	$\top \to \varphi \vee \neg\varphi$	$xCy \to y \leq x$
ν	contardiction	$\varphi \wedge \neg\varphi \to \bot$	xCx
δ	double negation	$\neg\neg\varphi \to \varphi$	$\exists y(xCy \wedge \forall z(yCz \to z \leq x))$
π	contraposition	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	$Rxyz \wedge z'Cz \to (\exists y')(y'Cy \wedge Rxz'y')$

In what follows, the particular choice of axioms from the table above does not play an important role. Therefore, from now on, we denote $\mathcal{L}$ a calculus consisting of the axiom schemes and rules for $\mathcal{L}_0^K$ extended with some axiom schemes from the table above, and possibly with some axioms and rules of the Lemma 2.8. We denote $\mathcal{F}^S_{\mathcal{L}}$ the class of substructural frames satisfying the frame conditions corresponding to the axioms present in $\mathcal{L}$.

It is common to spell out the added axioms using their corresponding signs. For example, if we consider the fragment of the language without negation, then $\mathcal{L}_a$ is the epistemic extension of associative commutative distributive full Lambek calculus, and $\mathcal{L}_{awc}$ is the epistemic extension of intuitionistic logic. $\mathcal{L}_{ac}$ plus double negation and contraposition corresponds to the the epistemic extension of relevant logic R, $\mathcal{L}_{awc}$ plus double negation and contraposition corresponds to the epistemic extension of classical logic. Using the table above one can construct various epistemic logics and their frame semantics. We give some further examples concerning intuitionistic and classical logics as the background logics.

Example 3.7 (Intuitionistic epistemic logic)

(i) Simplest way of arriving at intuitionistic epistemic logic is given by $\mathcal{L}_{\mathrm{awc}}$ and its corresponding class of frames.

(ii) Let us see how to fit in the standard semantics of intuitionistic logic. Consider any poset $(X,\leq)$ (an intuitionistic frame), put $L=X$, let S to be any monotone relation satisfying $S\subseteq\leq$, and define the remaining relations as follows:

$$\begin{aligned} Rxyz &\quad \text{iff} \quad x\leq z \text{ and } y\leq z \\ Cxy &\quad \text{iff} \quad \exists z(x\leq z \text{ and } y\leq z) \end{aligned}$$

Such frames are frames for intuitionistic logic (it is an interesting exercise to see that the interpretation of implication and negation given by the defined relations coincides with the standard intuitionistic interpretations of the two connectives). They are equipped with the source relation S. The epistemic modality K is a backward-looking diamond. The frames validate intuitionistic logic plus axioms and rules for the K modality from Definition 3.3. The modality is not trivial ($\varphi\nvdash K\varphi$), and neither it commutes with the conjunction nor distributes to the implication.

(iii) Consider now the poset $(X,\leq)$ to be moreover a rooted tree with the root r. Put rSx for all $x\in X$. Thus, the root r is a universal source. Such a frame can be seen as depicting possible exploitations of the universal source.

In this class of frames, as can be checked e.g. via the frame conditions of Lemma 2.8, K commutes with conjunction, distributes to implication, positive introspection axiom becomes valid, as well as negative introspection axiom. To see the last one, assume that $x\Vdash\neg K\varphi$. Then no state compatible with x validates $K\varphi$, and since there are such states (namely any state $\leq$-comparable with x is compatible with x) and r is a source for them, it follows that $r\nVdash\varphi$. Then no state validates $K\varphi$, and in particular no state compatible with r does so. Thus $r\Vdash\neg K\varphi$ and therefore $x\Vdash K\neg K\varphi$.

Example 3.8 (What about classical logic?)

Consider frames validating $\top\vdash\varphi\vee\neg\varphi$. The corresponding frame condition together with the symmetry of C entail that xCy implies $x=y$ (together with the scheme $\varphi\wedge\neg\varphi\vdash\bot$ and the corresponding condition xCx we obtain that C is the equality). By the condition that $S\subseteq C$, also xSy implies $x=y$. Therefore, the frames for classical logic can be meaningfully equipped only with somewhat uninteresting source relation—the only source for a state can be the state itself. Still the modality does not trivialize ($\varphi\nvdash K\varphi$). The positive introspection axiom becomes valid, while the negative introspection may fail (because there can be states that are not sources for themselves, and therefore have no source).

THEOREM 3.9 (SOUNDNESS)

$\Gamma \vdash_{\mathcal{L}} \psi$ implies $\Gamma \vDash_{\mathcal{F}^S_{\mathcal{L}}} \psi$.

PROOF. The proof of the soundness of the non-modal part can be performed by a routine induction on the proofs, using the correspondence of the axioms to the respective frame conditions. The soundness of the modal axioms has been established in Lemma 2.7. ■

4 Completeness

We prove that the axiomatization of $\mathcal{L}$ is strongly complete with respect to the class of corresponding epistemic frames $\mathcal{F}^S_{\mathcal{L}}$, i.e. $\Gamma \vDash_{\mathcal{F}^S_{\mathcal{L}}} \varphi$ implies $\Gamma \vdash_{\mathcal{L}} \varphi$. We shall proceed via the standard *canonical model* construction. Since the construction is based on the one for the background substructural logic $\mathcal{L}_0$ included in [15] (which also applies to its extensions with monotone modalities like e.g. a backward-looking diamond) we only recall the necessary ingredients and concentrate on the particulars needed to ensure that the construction works for the epistemic extension $\mathcal{L}$ as well. Namely, we need to ensure that the canonical relations S and C and $\leq$ relate in the proper way, and therefore the canonical model of $\mathcal{L}$ is based on a frame validating the epistemic axioms of truth and consistency—showing that the axioms are *canonical*.

The construction of canonical model of a logic $\mathcal{L}$ uses prime theories of the logic. A set of formulas Γ is called a $\mathcal{L}$*-prime theory* if for all formulas φ and ψ if $\Gamma \vdash_{\mathcal{L}} \varphi \vee \psi$ then either $\varphi \in \Gamma$ or $\psi \in \Gamma$. Thus, in particular, if Γ is a $\mathcal{L}$-prime theory and $\Gamma \vdash_{\mathcal{L}} \varphi$, then, by $\Gamma \vdash_{\mathcal{L}} \varphi \vee \varphi$ we obtain $\varphi \in \Gamma$. Thus prime theories are deductively closed. Note that if $\bot \in \Gamma$ then for all formulas φ, $\Gamma \vdash \varphi$, and therefore Γ coincides with the entire set of formulas. We use the prime theories over $\mathcal{L}$ as the canonical set of points. For the sake of proving completeness, we are only interested in prime theories that do not prove (or equivalently contain) some formula. Therefore, we will consider only prime theories, which do not coincide with the entire set of formulas, as the canonical set of points. We will call such prime theories *proper*. Note that the canonical model defined this way may contain points in which $\mathtt{t}$ is not included, thus it contains non-logical situations. It is not surprising since e.g. to violate the weakening axiom we need some non-logical situations included. Prime theories can be built over certain sets of formulas arising as left parts of the so-called *pairs*:

DEFINITION 4.1
Let Γ, Δ denote sets of formulas. $P = \langle \Gamma, \Delta \rangle$ is called a $\mathcal{L}$*-pair* whenever $\Gamma \nvdash_{\mathcal{L}} \Delta$. It is called a *full pair*, if $\Gamma \cup \Delta$ is the whole language. A pair $< \Gamma', \Delta' >$ extends a pair $< \Gamma, \Delta >$ if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$.

The consequence relation $\vdash$ of Definition 3.3 is pair extension acceptable, see [15, Definition 5.11 p. 92]. Therefore Pair Extension Theorem [15, Theorem 5.17 p. 94] applies.

THEOREM 4.2 (PAIR EXTENSION THEOREM)
Let $P = \langle \Gamma, \Delta \rangle$ be a $\mathcal{L}$-pair. Then there exists a full pair $P' = \langle \Gamma', \Delta' \rangle$ extending P.

In a full pair, Γ' is always a $\mathcal{L}$-prime theory [15, p. 93]. Also observe that whenever $\Delta \neq \emptyset$ in the original pair, we obtain by extension to a full pair a proper prime theory Γ' which does not contain $\bot$. In the sequel, we will apply the Pair Extension Theorem to construct prime theories satisfying particular properties. We will be interested in proper prime theories. To ensure that $\Delta \neq \emptyset$ we will implicitly assume that it contains $\bot$.

THEOREM 4.3 (STRONG COMPLETENESS)
The axiomatization $\mathcal{L}$ is strongly complete with respect to the class of corresponding epistemic frames $\mathcal{F}^S_{\mathcal{L}}$.

PROOF. We use the standard Henkin-style construction of a canonical model, taken e.g. from [15, Sections 11.3 and 11.4]. We define the canonical model $M_{\mathcal{L}}$, parametric in $\mathcal{L}$. We must pay attention to the accessibility relation $S_{\mathcal{L}}$ defined on the canonical model, to verify that its underlying frame satisfies the frame conditions for $\mathcal{F}^S_{\mathcal{L}}$, and hence it is an epistemic frame.

Following the proof in [15], we take the points $W_{\mathcal{L}}$ in the canonical frame to be all the proper prime theories of the logic $\mathcal{L}$, and define the canonical frame to be $F_{\mathcal{L}} = (W_{\mathcal{L}}, L_{\mathcal{L}}, \leq_{\mathcal{L}}, C_{\mathcal{L}}, R_{\mathcal{L}})$ with the canonical relations defined in the standard way as follows:

- $L_{\mathcal{L}} = \{x \mid \mathtt{t} \in x\}$
- $x \leq_{\mathcal{L}} y$ iff $x \subseteq y$
- $R_{\mathcal{L}} = \{(xyz) \mid \forall \varphi, \psi\ (\varphi \to \psi \in x$ and $\varphi \in y$ implies $\psi \in z)\}$
- $C_{\mathcal{L}} = \{(xy) \mid \forall \varphi\ (\neg\varphi \in x$ implies $\varphi \notin y)\}$
- $S_{\mathcal{L}} = \{(xy) \mid \forall \varphi\ (\varphi \in x$ implies $K\varphi \in y)\}$

Adding the canonical valuation V defined as

$$V(p) = \{x \mid p \in x\}$$

we get the canonical model $M_{\mathcal{L}}$. It is immediate from the definition that membership in the prime theory satisfies the semantic conditions for $\wedge$ and $\vee$, and half of the semantic conditions for the implication, negation and for the modality K. We have the following:

- if $\varphi \to \psi \in x$, then if $R_{\mathcal{L}}xyz$, and $\varphi \in y$, then $\psi \in z$
- if $\neg\varphi \in x$, then if $xC_{\mathcal{L}}y$, then $\varphi \notin y$
- if $\varphi \in x$, then if $xS_{\mathcal{L}}y$, then $K\varphi \in y$.

To ensure that we have the converse of these conditions as well, i.e. to give a proof of valuation lemma, we appeal to [15, Wittness Lemma 11.29 p. 255] and [15, Lemma 11.30], based on the Pair Extension Theorem [15, Theorem 5.17 p. 94]. The proof for $R_{\mathcal{L}}$ and $C_{\mathcal{L}}$ is standard [15, pp. 256, 261], but the proof for $S_{\mathcal{L}}$ we include here.

LEMMA 4.4 (VALUATION LEMMA)
$x \Vdash \varphi$ iff $\varphi \in x$

PROOF. We will show that $x \Vdash K\varphi$ iff $K\varphi \in x$, the rest of the proof is similar and can be extracted from [15].

From left to right: $x \Vdash K\varphi$, thus there is $sS_{\mathcal{L}}x$ satisfying φ. From the definition of $S_{\mathcal{L}}$, $K\varphi \in x$.

From right to left (witnessing lemma for K): Let us assume $K\varphi \in x$, we need an $s \in W_{\mathcal{L}}$ such that $sS_{\mathcal{L}}x$ and $\varphi \in s$. $P = \langle\{\varphi\}; \{\psi : K\psi \notin x\}\rangle$ is a pair. Because if not, then $\varphi \vdash \psi_1 \vee \ldots \vee \psi_n$, hence $K\varphi \vdash K(\psi_1 \vee \ldots \vee \psi_n)$ (monotonicity of K), and $K\varphi \vdash K\psi_1 \vee \ldots \vee K\psi_n$ (distribution of K over disjunction). But as $K\varphi \in x$, then $K\psi_1 \vee \ldots \vee K\psi_n \in x$ (as x is a theory), hence $K\psi_i \in x$ (as x is prime), which is a contradiction with the definition of P.
According to Pair Extension Theorem we can extend P to a full pair $P' = \langle s, r\rangle$. It follows, that s is prime and proper. It remains to show that $sS_{\mathcal{L}}x$. Assume $\alpha \in s$. If $K\alpha \notin x$, then $\alpha \in r$ (definition of P), so $\alpha \notin s$. Contradiction. ∎

We have shown that the canonical model satisfies the general conditions for an epistemic model. It remains to show that the canonical frame falls within the class of epistemic frames, checking that the canonical relations satisfy the required conditions.

LEMMA 4.5
$F_{\mathcal{L}} \in \mathcal{F}^S_{\mathcal{L}}$

PROOF. It is almost immediate that $R_{\mathcal{L}}, C_{\mathcal{L}}$ defined above satisfy the monotonicity conditions (1) and (4) from Definition 2.1. (In [15] the relations satisfying these conditions are called 'plump'). We skip these cases as well here and refer to [15] where they can be easily extracted from the general proof. Similarly $S_{\mathcal{L}}$ satisfies the monotonicity condition (8) from Definition 2.2:

Suppose e.g. that $sS_{\mathcal{L}}x$ and $s' \subseteq s$. Then any $\varphi \in s'$ is also contained in s and therefore $K\varphi \in x$ as required. The remaining case is similar.

For the frame conditions forced by axioms in $\mathcal{L}$ we can again refer to the proof in [15]. We show for an illustration that $R_{\mathcal{L}}$ is commutative and that $C_{\mathcal{L}}$ is symmetric:

Suppose $R_{\mathcal{L}}xyz$. To prove $R_{\mathcal{L}}yxz$ assume $\varphi \to \psi \in y$ and $\varphi \in x$. Then by the rules of the calculus also $\mathtt{t} \to (\varphi \to \psi) \in y$ and by the exchange scheme $\varphi \to (\mathtt{t} \to \psi) \in y$. By $R_{\mathcal{L}}xyz$ we obtain $\mathtt{t} \to \psi \in z$ and therefore $\psi \in z$.

Suppose $xC_{\mathcal{L}}y$. To prove $yC_{\mathcal{L}}x$ assume $\neg\varphi \in y$. If φ was in x, then by $\varphi \vdash \neg\neg\varphi$ also $\neg\neg\varphi \in x$. By $xC_{\mathcal{L}}y$, $\neg\varphi \notin y$ - a contradiction. Thus $\varphi \notin x$ as desired.

Next we concentrate on the truth set $L_{\mathcal{L}}$ and the relation $S_{\mathcal{L}}$. To make clear the situation of logical states we show that $L_{\mathcal{L}}$ satisfies the required conditions. First observe that $L_{\mathcal{L}}$ is clearly an upper set w.r.t. inclusion. We show that it satisfies the condition (3).

Suppose $R_{\mathcal{L}}xyz$ and $x \in L_{\mathcal{L}}$. To show $y \subseteq z$ suppose $\varphi \in y$. Since $\mathtt{t} \in x$ and $\mathtt{t} \vdash \varphi \to \varphi$, we have $\varphi \to \varphi \in x$. Since $R_{\mathcal{L}}xyz$ and $\varphi \in y$, we obtain $\varphi \in z$ as desired.

For the converse suppose $y \subseteq z$. We have to find a prime theory x such that $\mathtt{t} \in x$ and $R_{\mathcal{L}}xyz$ holds. First observe that $P = \langle \{\mathtt{t}\}, \{\varphi \to \psi \mid \varphi \in y, \psi \notin z\} \rangle$ is a pair: if not, $\mathtt{t} \vdash \bigvee_{i\in I}(\varphi_i \to \psi_i)$ for some disjunction of implications from the right set in the pair P. Then we reason as follows, using the provable monotonicity of $\otimes$, its distribution over $\vee$, and neutrality of $\mathtt{t}$:

$$\mathtt{t} \vdash \bigvee_{i\in I}(\varphi_i \to \psi_i)$$

$$\bigwedge_{j\in I}\varphi_j \vdash \bigwedge_{j\in I}\varphi_j \otimes \bigvee_{i\in I}(\varphi_i \to \psi_i)$$

$$\bigwedge_{j\in I}\varphi_j \vdash \bigvee_{i\in I}((\bigwedge_{j\in I}\varphi_j)\otimes(\varphi_i \to \psi_i))$$

Then, since for each $i \in I$ it holds that $(\bigwedge_{j\in I}\varphi_j)\otimes(\varphi_i \to \psi_i) \vdash \psi_i$, we obtain $\bigwedge_{j\in I}\varphi_j \vdash \bigvee_{i\in I}\psi_i$. But since all $\varphi_j \in y$, we have $\bigvee_{i\in I}\psi_i \in y$ and thus some $\psi_i \in y$, which is a contradiction.

According to Pair Extension Theorem we can extend P to a full pair $P' = \langle x, u \rangle$ where x is a prime theory. Moreover, we have $\mathtt{t} \in x$ and $x \in L_{\mathcal{L}}$. $R_{\mathcal{L}}xyz$ holds immediately from the definition of P: if $\varphi \in y$ and $\psi \notin z$ then, from the definition of P, $\varphi \to \psi \in u$ and thus $\varphi \to \psi \notin x$.

We show that the canonical relation $S_{\mathcal{L}}$ satisfies the frame condition (6):

Suppose $sS_{\mathcal{L}}x$. To show that $s \leq x$, reason as follows: if $\varphi \in s$, then $K\varphi \in x$ (by the definition of $S_{\mathcal{L}}$) and by factivity (any prime theory containing $K\varphi$ is closed under the truth scheme because $K\varphi \vdash \varphi$), $\varphi \in x$ too. Since the order on the canonical model is the inclusion, we have $s \leq x$.

We show that the canonical relation $S_{\mathcal{L}}$ satisfies the frame condition (7):

Suppose $sS_{\mathcal{L}}x$. We show that $xC_{\mathcal{L}}s$. Reason as follows: If $\neg\varphi \in x$, then suppose φ were in s. Then since $sS_{\mathcal{L}}x$, we have $K\varphi \in x$. But we have $\neg\varphi \in x$. Thus, $\neg\varphi \wedge K\varphi \in x$, which by the scheme of consistency is impossible (x is a proper prime theory, and no proper prime theories contain $\perp$). So, if $\neg\varphi \in x$, then $\varphi \notin s$, and hence, $xC_{\mathcal{L}}s$. From symmetry $sC_{\mathcal{L}}x$ as desired.

Next we make sure that $S_{\mathcal{L}}$ satisfies the condition (8):

Suppose $sS_{\mathcal{L}}x$ and $x \subseteq x'$. We need a prime theory s' such that $s \subseteq s'$ and $sS_{\mathcal{L}}x'$. We claim we can take $s' = s$. We show $sS_{\mathcal{L}}x'$: suppose $\varphi \in s$, then by $sS_{\mathcal{L}}x$ and definition of $S_{\mathcal{L}}$, $K\varphi \in x$. Since $x \subseteq x'$, we have $K\varphi \in x'$ as well. Similarly for $sS_{\mathcal{L}}x$ and $s'' \subseteq s'$ ■

Recall from Lemma 2.8 the characteristic frame conditions for the epistemic axioms of positive and negative introspection, necessitation rule and $\to, \wedge$ and $\otimes$ distributivity. The fact that if $\mathcal{L}$ contains these axioms or rules and $\mathcal{L}$ prime theories contain or are closed under them is not enough to derive that they are canonical—that they hold on their respective canonical frames. They clearly hold on the canonical *model* but it can be because of the canonical valuation.

- Canonical frame of $\mathcal{L}$ containing the necessitation rule satisfies the condition $(\forall x \in L_{\mathcal{L}})\ (\exists s \in L_{\mathcal{L}})(sS_{\mathcal{L}}x)$:
  Suppose $x \in L_{\mathcal{L}}$. Observe that in the logic with necessitation, $K\mathtt{t}$ is a theorem, therefore $\mathtt{t} \vdash K\mathtt{t}$ and thus $K\mathtt{t}$ is also present in all the prime theories containing $\mathtt{t}$ which are the logical states. Thus $K\mathtt{t} \in x$ and by the witnessing lemma there is a source $sS_{\mathcal{L}}x$ such that $\mathtt{t} \in s$, meaning $s \in L_{\mathcal{L}}$.
- Canonical frame of $\mathcal{L}$ containing the $\to$-distribution axiom (or equivalently $\otimes$-distribution 1 axiom) satisfies the condition $(R_{\mathcal{L}}xyz \wedge sS_{\mathcal{L}}x \wedge tS_{\mathcal{L}}y) \to \exists w(wS_{\mathcal{L}}z \wedge R_{\mathcal{L}}stw)$:
  Assume $R_{\mathcal{L}}xyz \wedge sS_{\mathcal{L}}x \wedge tS_{\mathcal{L}}y$. We construct a required w by pair extension theorem. First let us see that $P = \langle \{\alpha \otimes \beta \mid \alpha \in s, \beta \in t\}, \{\gamma \mid K\gamma \notin z\}\rangle$ is a pair: suppose for a contradiction that some $\bigwedge_i(\alpha_i \otimes \beta_i) \vdash \bigvee_j \gamma_j$. Since $\bigwedge_i \alpha_i \otimes \bigwedge_i \beta_i \vdash \bigwedge_i(\alpha_i \otimes \beta_i)$, we obtain $\bigwedge_i \alpha_i \otimes \bigwedge_i \beta_i \vdash \bigvee_j \gamma_j$, and by monotonicity also $K(\bigwedge_i \alpha_i \otimes \bigwedge_i \beta_i) \vdash K \bigvee_j \gamma_j$. By the $\otimes$-distribution' and $\vee$ distribution we have $\bigwedge_i K\alpha_i \otimes \bigwedge_i K\beta_i \vdash \bigvee_j K\gamma_j$. Since every $K\alpha_i \in x$ and $K\beta_i \in y$ and $R_{\mathcal{L}}xyz$, we obtain some $K\gamma_j \in z$ by z being prime. A contradiction.
  Therefore, P it is a pair and as such it can be extended to a full pair $P' = \langle w, u\rangle$, where $(w \cup u)$ is the whole language and w is the required prime theory. It follows from the definition of the pair P that indeed $wS_{\mathcal{L}}z$ and $R_{\mathcal{L}}stw$.
- Canonical frame of $\mathcal{L}$ containing the $\otimes$-distribution 2 axiom satisfies the condition $(R_{\mathcal{L}}uvs \wedge sS_{\mathcal{L}}z) \to \exists x \exists y(uS_{\mathcal{L}}x \wedge vS_{\mathcal{L}}y \wedge R_{\mathcal{L}}xyz)$:
  Assume $R_{\mathcal{L}}uvs \wedge sS_{\mathcal{L}}z$. First we define two theories as follows—x' is the deductive closure of the set $\{K\alpha \mid \alpha \in u\}$ and y' is the deductive closure of the set $\{K\beta \mid \beta \in v\}$. Observe that for any $\varphi \in x'$ and $\psi \in y'$ we have $\varphi \otimes \psi \in z$: if $\varphi \in x'$ then some $\bigwedge(K\alpha) \vdash \varphi$, thus $(K \bigwedge \alpha) \vdash \varphi$. Similarly some $(K \bigwedge \beta) \vdash \psi$ and therefore $(K \bigwedge \alpha) \otimes (K \bigwedge \beta) \vdash \varphi \otimes \psi$. By the $\otimes$-distributivity $K(\bigwedge \alpha \otimes \bigwedge \beta) \vdash \varphi \otimes \psi$. By $\bigwedge \alpha \in u$ and $\bigwedge \beta \in v$ and $R_{\mathcal{L}}uvsS_{\mathcal{L}}z$ we obtain $\varphi \otimes \psi \in z$.
  Now we may apply the [15, Squeeze lemma 11.32] as follows: first extend x' to a prime theory x using a pair $\langle x', \{\gamma \mid (\exists \delta \in y')\ \gamma \otimes \delta \notin z\}\rangle$. Then we extend y' to a prime theory y using a pair $\langle y', \{\gamma \mid (\exists \delta \in x)\ \delta \otimes \gamma \notin z\}\rangle$. We leave the remaining details to the reader.

- Canonical frame of $\mathcal{L}$ containing the $\wedge$-distribution axiom satisfies the condition $sS_{\mathcal{L}}x \wedge tS_{\mathcal{L}}x \to \exists v(vS_{\mathcal{L}}x \wedge s \leq v \wedge t \leq v)$:
  Assume $sS_{\mathcal{L}}x \wedge tS_{\mathcal{L}}x$. We construct v by pair extension theorem. Observe that $P = \langle s \cup t, \{\alpha \mid K\alpha \notin x\}\rangle$ is a pair: let us have, for a contradiction, for some $\bigwedge \varphi \in s$ and $\bigwedge \psi \in t$ that $\bigwedge \varphi \wedge \bigwedge \psi \vdash \bigvee \alpha$. Then by monotonicity, $\vee$-distribution and $\wedge$-distribution we also have $\bigwedge K\varphi \wedge \bigwedge K\psi \vdash \bigvee K\alpha$ with all the $K\varphi \in x$ and $K\psi \in x$, contradicting the fact that no $K\alpha \in x$. Extending the pair P to a full pair $P' = \langle v, w\rangle$ yields the required prime theory v which clearly satisfies $s \subseteq v$ and $t \subseteq v$ and $vS_{\mathcal{L}}x$.
- Canonical frame of $\mathcal{L}$ containing the positive introspection axiom satisfies the condition $sS_{\mathcal{L}}x \to \exists y(sS_{\mathcal{L}}yS_{\mathcal{L}}x)$:
  Suppose that $sS_{\mathcal{L}}x$. Then $P = \langle \{K\alpha \mid \alpha \in s\}, \{\beta \mid K\beta \notin x\}\rangle$ is a pair: if not, then some $\bigwedge K\alpha \vdash \bigvee \beta$. Then also $K \bigwedge \alpha \vdash \bigvee \beta$, and therefore by monotonicity $KK \bigwedge \alpha \vdash K \bigvee \beta$. By the positive introspection $K \bigwedge \alpha \vdash K \bigvee \beta$ and consequently $K \bigwedge \alpha \vdash \bigvee K\beta$, with $\bigwedge \alpha \in s$ by s being a theory. Therefore, $K \bigwedge \alpha \in x$ contradicting the fact that no $K\beta \in x$. Extending the pair P to a full pair $P' = \langle y, w\rangle$ yields a prime theory y with $sS_{\mathcal{L}}yS_{\mathcal{L}}x$.
- Canonical frame of $\mathcal{L}$ containing the negative introspection axiom satisfies the condition $\exists s(sS_{\mathcal{L}}x \wedge \forall u \forall v(uS_{\mathcal{L}}v \wedge sC_{\mathcal{L}}v \to \exists y \exists z(yS_{\mathcal{L}}z \wedge xC_{\mathcal{L}}z)))$:
  Assume a prime theory x is given. We construct s using the pair extension theorem as follows: we show that $P = \langle \{\neg K\alpha \mid \neg K\alpha \in x\}, \{\beta \mid K\beta \notin x\}\rangle$ is a pair. If it was not a pair, then some $\bigwedge(\neg K\alpha) \vdash \bigvee \beta$. Since $\bigwedge(\neg K\alpha)$ is provably equivalent with $\neg K(\bigvee \alpha)$, we obtain $\neg K(\bigvee \alpha) \vdash \bigvee \beta$ and by monotonicity $K\neg K(\bigvee \alpha) \vdash K \bigvee \beta$. Applying negative introspection and $\vee$-distributivity we get $\neg K(\bigvee \alpha) \vdash \bigvee K\beta$ where $\neg K(\bigvee \alpha) \in x$ by x being a theory containing $\bigwedge(\neg K\alpha)$. It contradicts the fact that no $K\beta$ is in x. Extending the pair P to a full pair $P' = \langle s, w\rangle$ we obtain a prime theory s with $sS_{\mathcal{L}}x$ which moreover satisfies the rest of the required properties (we leave it to the reader to check those).

This finishes the proof of Theorem 4.3. ■

4.1 Finite model property

A natural question to ask is how the extension of the underlying logic with the epistemic modality and the two axioms affects decidability of the underlying substructural logic. We give a particular answer to this question. We present an argument based on a filtration of the canonical model to show that the weakest epistemic logic we consider in this article (which is the distributive commutative non-associative full Lambek with a negation and K modality) given by the axiomatization $\mathcal{L}_0^K$ has the finite model property, and is thus decidable.

The underlying logic $\mathcal{L}_0$ is decidable. This can be seen e.g. from a smallest filtration of the canonical model as shown in [15, Theorem 14.11]. The argument in [15, Theorem 14.11] applies to a modal extension with a backward-looking diamond as well, but with no additional axioms for the diamond, and therefore no additional frame properties. Moreover, since transitivity of a relation is not automatically preserved by the smallest filtration, the argument treats the preorder on the filtration model as the discrete one (this can be always done and then the monotonicity of all the relations trivially follows.) However, we require the source relation to be a part of the preorder, and therefore we cannot hope to do with the discrete preorder and make the filtration still work this way. Therefore, we have to modify the argument.

THEOREM 4.6
The logic $\mathcal{L}_0^K$ has the finite model property, and is thus decidable.

PROOF. Assume we want to refute an invalid formula φ in $\mathcal{L}_0^K$. We define a set of formulas Φ to be the set of all subformulas of φ, plus for each subformula of φ which is of the form $\neg\alpha$ we also include the formulas $K\alpha$ and $K\neg\alpha$, plus we include the constant $\mathtt{t}$. The set Φ is finite.

We define a filtration of the canonical model $M=(W,R,L,C,S,\Vdash)$, using the inclusion of the corresponding sets of formulas from Φ as the preorder, and ensuring all the relations are monotone and so that the source relation relates to the preorder and compatibility in the proper way.

Let $\sim$ be the equivalence relation on W generated by the set Φ and defined as follows:

$$x \preceq y \ \text{ iff } \ (\forall \alpha \in \Phi)\,(x \Vdash \alpha \Rightarrow y \Vdash \alpha)$$
$$x \sim y \ \text{ iff } \ x \preceq y \ \text{ and } \ y \preceq x$$

Since Φ is finite, $\sim$ has only finitely many equivalence classes. Now we define the filtration model as follows:

- $W^\Phi = \{[x] \mid x \in W\}$
- $L^\Phi = \{[x] \mid (\exists x' \preceq x)\, x' \in L\}$
- $[x] \leq^\Phi [y]$ iff $x \preceq y$
- $[x]C^\Phi[y]$ whenever $(\exists x' \succeq x)(\exists y' \succeq y)\, x'Cy'$
- $([x](S^\Phi \cap C^\Phi)[y]$ and $[y]C^\Phi[x])$ iff $(\exists x' \succeq x)(\exists y' \preceq y)\, x'(S\cap C)y'$
- $R^\Phi[x][y][z]$ iff $(\exists x' \succeq x)(\exists y' \succeq y)(\exists z' \preceq z)\, Rx'y'z'$

The $\preceq$ is in general a preorder on the original model, but $\leq^\Phi$ becomes a partial order on the equivalence classes. The definition takes care of all the defined relation being monotone in the proper way with respect to the new partial order. The fourth and fifth item means that we define C^Φ using both C alone and $C\cap S$ from the original model. Using C alone would not be enough to ensure that the source relation will be contained in the compatibility relation.

Let us see that the relations satisfy the required properties of Definition 2.1. They are by definition monotone w.r.t. $\preceq$ in the right way, C^Φ is symmetric, R^Φ is commutative, and S^Φ is automatically a subset of C^Φ (being defined via $S^\Phi \cap C^\Phi$), and finally S^Φ is a subset of the preorder $\leq^\Phi$: indeed, if $[x]S^\Phi[y]$, then $x \preceq x'Sy' \preceq y$. But $x'Sy'$ implies $x' \leq y'$ and by persistency also $x' \preceq y'$, therefore $x \preceq y$ as required. Observe that $[u] \in L^\Phi$ if and only if $u \in L$. It is not hard to see that R^Φ and L^Φ relate properly: assume first there is $[u] \in L^\Phi$ with $R^\Phi[u][x][y]$. Then we know that $Rux'y'$ for some $x' \succeq x$ and $y' \preceq y$. But since $u \in L$, we know that $x' \leq y'$ and, therefore, $x' \preceq y'$ and $x \preceq y$ as required.

Assume now that $x \preceq y$. Consider the following pair $< x \cap \Phi; \{\alpha \mid \alpha \notin y\} >$. It is indeed a pair, since if $\bigwedge \varphi \vdash \bigvee \alpha$ with all $\varphi \in x \cap \Phi$, we would also have all $\varphi \in y$ and from y being a prime theory some $\alpha \in y$, which is not possible. We extend the pair to obtain a prime theory x' with $x \preceq x'$. Clearly $x' \subseteq y$, therefore there exists some $u \in L$ with $Rux'y$. It follows that $R^\Phi[u][x][y]$ as required.

It holds that $x \leq y$ implies $[x] \leq^\Phi [y]$, xCy implies $[x]C^\Phi[y]$, and analogously for R and S.

We finish the definition of the filtration letting the valuation on the filtration model to be, for each $p \in \Phi$ and each $[x] \in W^\Phi$

$$[x] \Vdash p \ \text{ iff } \ x \Vdash p.$$

The valuation is clearly persistent. Now we have to prove that the model we have just defined is indeed a filtration of the original model:

LEMMA 4.7
For all $\alpha \in \Phi$: $M, x \Vdash \alpha$ if and only if $M^{\Phi}, [x] \Vdash \alpha$

REMARK 4.8
We start with a remark that the obvious map $[\ .\ ] : W \to W^{\Phi}$ is in general not a frame morphism, thus we cannot simply use the fact that frame morphisms preserve formulas here.

To see this, recall the last item in Definition 2.9 and consider the following situation in the canonical model. We assume that $\Phi = \{q, \mathtt{t}\}$. Consider pairs $P_1 = <\neg q; q, \mathtt{t}>$ and $P_2 = <q; \neg q>$. Extend P_1 to a full pair obtaining a proper prime theory x, and extend P_2 to a full pair obtaining a proper prime theory y. It holds that $[x]C^{\Phi}[y]$: since $\neg q \notin y$, we know that there is a prime theory $z \ni q$ with zCy. Since $x \cap \Phi = \emptyset$, we know $x \preceq z$ trivially holds. Therefore, by definition of the filtration, $[x]C^{\Phi}[y]$. But there is no $y' \succeq y$ with $y'Cx$, because every such y' contains q while x contains $\neg q$.

PROOF. We prove Lemma 4.7 by induction on α. If α is an atom, the claim holds by definition. The cases for $\wedge$ and $\vee$ are easy, they use the induction hypothesis and the closure of Φ under subformulas.

- Let $\alpha = \mathtt{t}$. $[x] \vdash \mathtt{t}$ if and only if $[x] \in L^{\Phi}$ if and only if
$$L \ni x' \preceq x.$$
by $x' \in L$ we know that $\mathtt{t} \in x'$. By $\mathtt{t} \in \Phi$ and persistence $\mathtt{t} \in x$, which in turn holds if and only of $x \Vdash \mathtt{t}$.
- Let α be $\neg\beta$. Assume $x \Vdash \neg\beta$, we show that $[x] \Vdash \neg\beta$. Suppose $[x]C^{\Phi}[y]$ and for a contradiction assume $[y] \Vdash \beta$.
We have to distinguish three cases—three types of a situation in the canonical model resulting in $[x]C^{\Phi}[y]$ being included in the closure:
$$\text{(i) } x \preceq x'Cy' \succeq y$$
Since $[y] \Vdash \beta$, by the induction hypothesis $y \Vdash \beta$ and by $\beta \in \Phi$ we obtain $y' \Vdash \beta$, thus $x' \nVdash \neg\beta$ and by a similar reasoning $x \nVdash \neg\beta$, a contradiction.
$$\text{(ii) } x \preceq x' (S \cap C) y' \preceq y$$
By $x \Vdash \neg\beta$ and $\neg\beta \in \Phi$ we obtain $x' \Vdash \neg\beta$. By $x'Sy'$ and $K\neg\beta \in \Phi$ we obtain $y' \Vdash K\neg\beta$. Therefore, $y \Vdash K\neg\beta$, but also by the induction hypothesis $y \Vdash \beta$, which by the consistency scheme is a contradiction.
$$\text{(iii) } y \preceq y' (S \cap C) x' \preceq x$$
By the assumption and the induction hypothesis $y \Vdash \beta$, and therefore $y' \Vdash \beta$ and $x \Vdash K\beta$. But $x \Vdash \neg\beta$, which again is a contradiction.

For the other direction, assume $[x] \Vdash \neg\beta$, we show that $x \Vdash \neg\beta$. Consider any y with xCy. Then $[x]C^{\Phi}[y]$ holds as well and by the assumption $[y] \nVdash \beta$. By the induction hypothesis $y \nVdash \beta$.
- Let α be $K\beta$. Assume $x \Vdash K\beta$. Thus, for some y with ySx we have that $y \Vdash \beta$. From ySx it follows that $[y]S[x]$ and by the induction hypothesis $[y] \Vdash \beta$, proving that $[x] \Vdash K\beta$.
Now assume that $[x] \Vdash K\beta$. Thus, there is some $[y]S^{\Phi}[x]$ such that $[y] \Vdash \beta$. Similarly as before, the situation looks as follows:
$$y \preceq y' (S \cap C) x' \preceq x.$$
By the induction hypothesis $y \Vdash \beta$ and thus $y' \Vdash \beta$ and $x' \Vdash K\beta$ and, therefore, also $x \Vdash K\beta$, since $K\beta \in \Phi$.

The remaining cases of $\rightarrow$ and $\otimes$ are similar and we leave them for the reader. ■

This finishes the proof of Theorem 4.6. ■

REMARK 4.9
It is worth noticing, that some of the frame conditions for epistemic axioms and rules are preserved by this particular filtration of canonical model. Namely, the density of S corresponding to the positive introspection axiom, and the existence of logical sources for logical states, corresponding to the necessitation rule (Lemma 2.8). Also the property $Rxxx$ corresponding to contraction, and the property $Rxzy \rightarrow x \leq z$ corresponding to weakening are preserved, as in the case of the smallest filtration.

On the other hand, some more involved frame properties, namely those corresponding to $\wedge$-distributivity, $\rightarrow$-distributivity, $\otimes$-distributivity, negative introspection or associativity are not preserved in general.

REMARK 4.10
Decidability of the distributive non-associative full Lambek calculus can be also proved using algebraic methods via finite embeddability property for residuated grupoids, as was done by Horčík and Haniková in [11], or using proof-theoretical methods as was done by Farulewski in [10]. The final model property for the distributive full Lambek calculus was proved by Kozak in [12].

5 Further work

The aim of our article was to generalize the idea of knowledge as information confirmed by a reliable source which was proposed in [4] to the most general class of substructural logics based on the relational semantics. There are some standard extensions of the framework we did not address in the article.

Our epistemic operator corresponds to what is called in the literature 'hard' or indefeasible knowledge—i.e. knowledge which does not change no matter what kind of new information the agent receives. This is forced by the requirement that a source of a given state should be consistent with that state (which together with the monotonicity conditions for the consistency relation entails mutual consistency of all sources). It is quite natural to consider as a next step situations when some sources carry information which contradicts the current state and hence the agent has to revise his/her knowledge if he/she wants to be consistent with that sources. This kind of knowledge, known as defeasible or 'soft' knowledge, is an essential part of dynamic approaches to epistemic logic and is extensively studied in the literature. Dynamic extension of our framework is far from being straightforward and we shall address it in our future work.

5.1 Multiagent version of the logic

The attempt to formalize an epistemic logic for the type of agents we have on mind would not make much sense if we did not consider its possibility to work for groups of agents as well, and did not try to extend it with some of the usual group-knowledge modal operators, among which the common knowledge modality is a prominent example. A multiagent setting with a working common knowledge modality could be a starting point to adding information and knowledge dynamics, like public announcement or various types of updates to our system. While a multiagent version of the logic is easily obtained and it retains all the properties discussed so far, like strong completeness, adding a common knowledge operator is not so easy.

We start with fixing a finite set of agents $I=\{1,\ldots,m\}$, and we extend the language with modalities K_i for each $i\in I$, reading the formula $K_i\varphi$ as 'the agent i knows φ'. An epistemic frame for $I=\{1,\ldots,m\}$ is now a basic substructural frame F extended with a finite set of source relations $\{S_i \mid i\in I\}$ labelled by the agents. Each of the relations is required to satisfy the conditions specified in Definition 2.2.

5.1.1 Common knowledge

By the duality theory, we can from each epistemic frame construct its complex algebra of the upward closed subsets of the frame (see e.g. [9]). It is in general a distributive lattice-ordered grupoid. The underlying lattice is then a complete distributive lattice. By Knaster–Tarski Fixed Point Theorem, each monotone map f on any complete lattice has the least fixpoint $\mu x.f(x)$ and the greatest fixpoint $\nu x.f(x)$. We can use this fact and define, semantically, fixpoints for various monotone formulas. One such example would be the following greatest fixed point

$$\nu x.\bigwedge_{i\in I} K_i(\varphi\wedge x),$$

expressing the common knowledge of φ. We extend the epistemic multiagent language with a new modality C. Given an epistemic model and writing $\|\varphi\|$ for the upperset $\{x \mid x\Vdash\varphi\}$ and $K_iU=\{x \mid (\exists y\in U)\ yS_ix\}$ for the operation corresponding dually to the relation K_i, we interpret the formula $C\varphi$ as follows:

$$\|C\varphi\|=\nu x.\bigcap_{i\in I} K_i(\|\varphi\|\cap x).$$

This definition says, that $C\varphi$ holds in x if and only if there is, upon unravelling the S_i relations (which may be reflexive on some points), an m-ary complete tree of sources running downwards from x (m is the number of agents).

The formula $C\varphi$ entails that each agent knows φ, and also for each finite chain of agents $\{i_0,\ldots,i_n\}$ (and for each n) it entails the formula $K_{i_0}\ldots K_{i_n}\varphi$. However, if the $\wedge$-distributivity is not present, the common knowledge of φ is not equivalent to the infinite conjunction of all these 'chain' formulas, neither to the infinite conjunction of formulas expressing the iterations 'everyone knows that everyone knows that ... φ'. It is easy to see that the monotonicity rule for C is valid, and that $C(\varphi\vee\psi)\rightarrow C\varphi\vee C\psi$ is a valid scheme.

The following axiom and rule express that $C\varphi$ is interpreted as the greatest fixed point of the formula $\bigwedge_{i\in I} K_i(\varphi\wedge x)$:

$$C\varphi\rightarrow\bigwedge_{i\in I} K_i(C\varphi\wedge\varphi) \qquad \frac{\alpha\vdash\bigwedge_{i\in I} K_i(\alpha\wedge\varphi)}{\alpha\vdash C\varphi}$$

For example, it can be proved that $C\varphi\rightarrow CC\varphi$. It is not hard to see that the axiomatics is sound for the interpretation given above. It is also not compact. However, it is unclear whether, at least in some cases of logics we consider, it is also weakly complete. First, unlike in the case of the μ-calculus, there is no general result available concerning completeness of this axiomatization for the fixed point extensions of substructural logics, and it seems to be a difficult task to prove such results. Secondly, since we only have a 'flat' fixed point of one particular scheme we may be still able to prove the completeness as it is done in the case of standard common knowledge operator logics, which is also not compact, but it can be shown to be weakly complete using finitary methods. However, in this particular case, our filtration argument from the previous section does not work. We leave the completeness of the common knowledge extensions for further research.

5.2 Additional modalities

We have not discussed the other modalities that could be defined on the basis of the epistemic relation S. Two of them behave well as they automatically yield persistent formulas—the backward-looking diamond an the forward-looking box modalities form an adjoint pair. We choose K to be a backward-looking diamond-like modality. Do the remaining companions have any reasonable meaning? The forward-looking box modality might be considered as a potential knowledge operator P—we look forward and ask for which states our current state is a source. As the current source potentially confirms all its data from the point of view of future states, all the data true in the current state will be confirmed in all the states, for which the current state is a source. The two modalities satisfy the following adjoint rule:

$$\frac{K\varphi \to \psi}{\overline{\varphi \to P\psi}}$$

A good reason to include the right adjoint to our framework comes from a proof-theoretical motivations—to come up with a decent proof system, like e.g. a display calculus, it is in general a good idea to have all the operators residuated.

From the other pair of modalities, the forward-looking diamond modality does not seem to have any intuitive interpretation. The box-like backward-looking modality seems to be the most interesting as it is a candidate for a stronger notion of knowledge. From the properties of K we know, that if ψ is known in a current state, there is no source which validates its negation (every source either confirms ψ or is 'opinionless' with respect to ψ. We might require a stronger confirmation—an uniform explicit confirmation by all sources (if any). Let us denote the operator $\blacksquare$. Strong knowledge is not persistent with respect to $\leq$, because it is interpreted by a relation which is monotone in the opposite way (and making it at the same time monotone in the other way, meaning we would require the sources be preserved downwards, has no clear epistemic motivation). Consider states $x, y, x \leq y$ such that y a source s, which is not a source for x. Then it can happen that there is a φ, which is uniformly confirmed in x, but $s \not\Vdash \varphi$, so φ is not uniformly confirmed in y. As one would expect, the monotonicity rule is valid for $\blacksquare$ (if $(\varphi \to \psi)$ is valid, then $(\blacksquare\varphi \to \blacksquare\psi)$) and it commutes with the conjunction $(\blacksquare(\varphi \wedge \psi) \to \blacksquare\varphi \wedge \blacksquare\psi)$.

5.3 Related work

In the literature, we can find several attempts to combine epistemic and substructural frameworks; however, they differ from our approach. Cheng [6] presents a modification of relevant logic with a motivation from the viewpoint of scientific discovery mechanism, which is an idea somewhat similar to our motivation. Wansing [18] introduces a relevant epistemic logic with strong negation. His approach is based on a combination of normal modal logic with the relevant one. Restall in [15] discusses modalities in a substructural framework in general.

There are also several quite recent publications in this area. Sequoiah-Grayson [17] works with epistemic interpretation of non-associative Lambek calculus; he models information flow between different states of one agent. Sedlár [16] proposes substructural epistemic logics by combining relational models for distributive substructural logics with (standard) Kripke models for normal modal logics. This framework is suitable for representation of non-monotonic information update and it is related to belief revision. Aucher [2] provides an interpretation of the ternary relation in the semantics of substructural logics in the terms of updates.

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